令和元年度 共同利用研究報告書

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下記の通り共同研究の報告をいたします.　　記

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7.参加者数	2 人 ＊1			

＊1 短期研究員は九大の共同研究者も含める.
研究集会 I, II, 短期共同研究は事務局から送った参加者データを元に記入.

8.本研究で得られた成果の概要(**成果報告書を別途要添付** 枚数は次頁参照)

今回の共同利用を用いた共同研究では、PDE などから定まるより複雑な力学系に対する Koopman 解析へのアプローチとして「一般化スペクトル理論」の研究を行った。Koopman 作用素は関数空間(無限次元線形空間)上の線形作用素であり、行列でいう固有値に対応する概念はスペクトルと呼ばれる。行列の場合と同様に、Koopman 作用素のスペクトルは力学系の主要なダイナミクス記述している重要な量である。ところが、行列の場合と著しく異なるのは、スペクトルの分布が連続的に広がる場合があり、そのようなスペクトルの解析は非常に難しいのが現状である。一般化スペクトル理論とはそのような連続的に広がるスペクトルの中から、ダイナミクスに本質に関わる部分を抽出する数学的なテクニックに 1 つであり、量子力学におけるトンネル効果や大自由度力学系における同期現象といった物理現象の解明において重要な貢献がある。一方、Koopman 作用素への一般化スペクトルの導入はあまり進んでいないが、データ駆動的な Koopman 作用素の解析において一般化スペクトルは重要な特徴量になると考えられるため、理論的な整備が望まれる。本研究では基本的な力学系である記号力学系に焦点を当ててその計算法について研究を行った。記号力学系は力学系の作用する底空間に解析構造が存在しないため、より代数的なアプローチが必要となる。本研究の成果は記号力学系の一般化スペクトル計算の基本方針の確立を行った。

平成30年度共同利用成果報告書

石川　勲

August 31, 2020

1 The generalized spectral theory

Let X be a locally convex Hausdorff topological vector space over $\mathbb{C}$ and X' its dual space. X' is a set of continuous anti-linear functionals on X. For $\mu \in X'$ and $f \in X$, $\mu(f)$ is denoted by $\langle \mu \,|\, f \rangle$. For any $a, b \in \mathbb{C}$, $f, g \in X$ and $\mu, \xi \in X'$, the equalities

$$\langle \mu \,|\, af + bg \rangle = \overline{a}\langle \mu \,|\, f \rangle + \overline{b}\langle \mu \,|\, g \rangle,$$
$$\langle a\mu + b\xi \,|\, f \rangle = a\langle \mu \,|\, f \rangle + b\langle \xi \,|\, f \rangle,$$

hold. The dual space X' is equipped with the weak dual topology (weak * topology) ; A sequence $\{\mu_j\} \subset X'$ is said to be weakly convergent to $\mu \in X'$ if $\langle \mu_j \,|\, f \rangle \to \langle \mu \,|\, f \rangle$ for each $f \in X$.

Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot\,,\,\cdot)$ such that X is a dense subspace of $\mathcal{H}$. Since a Hilbert space is isomorphic to its dual space, we obtain $\mathcal{H} \subset X'$ through $\mathcal{H} \simeq \mathcal{H}'$. The isomorphism $\mathcal{H} \simeq \mathcal{H}'$ is defined so that $\langle g \,|\, f \rangle = (g, f)$ when $g \in \mathcal{H}$.

Definitnion 1.1. If a locally convex Hausdorff topological vector space X is a dense subspace of a Hilbert space $\mathcal{H}$ and a topology of X is stronger than that of $\mathcal{H}$, the triplet

$$X \subset \mathcal{H} \subset X' \tag{1.1}$$

is called the *rigged Hilbert space* or the *Gelfand triplet*.

Let T be a linear operator densely defined on $\mathcal{H}$. The (Hilbert) adjoint T^* of T is defined through $(Tf, g) = (f, T^*g)$ as usual. If T^* is continuous on X, the dual operator $T' : X' \to X'$ of T^* defined through

$$\langle T'\mu \,|\, f \rangle = \langle \mu \,|\, T^*f \rangle, \quad f \in X, \, \mu \in X'$$

is also continuous on X' with respect to the weak dual topology. We can show the equality $T'g = Tg$ for any $g \in X$, which implies that T' is an extension of T. To define a generalized spectrum set of T, we further assume that X is a quasi-complete barreled space (see Tréves [5] for the definition), which is used to justify an integration of X'-valued complex functions [3]. Any Fréchet spaces, Banach spaces and Montel spaces satisfy this condition.

Let $\rho(T)$ be a resolvent set of T; the resolvent operator $R_\lambda = (\lambda - T)^{-1}$ is a continuous operator on $\mathcal{H}$ when $\lambda \in \rho(T)$ and is holomorphic on $\rho(T)$.

2 Generalized spectrums of one-sided shifts

In this section, we provide several computational results of generalized spectrums for several types for one-sided shits.

In this subsection, we describe an idea and detailed computations of the generalized spectrum of the one-sided shift. Here, we mainly deal with the one-sided full 2-shift, namely, $\Sigma = \{0,1\}^{\mathbb{N}}$ case. We treat a general case in the following subsection.

First, we fix several notations. let μ_0 be a measure on the set $\{0,1\}$ defined by $\mu_0(\{0\}) = \mu_0(\{1\}) = 1/2$, and define a measure μ^+ on Σ^+ by $\mu_0^{\otimes\mathbb{N}}$ as a product measure of μ_0's. We note that μ^+ is an invariant measure with respect to S, and thus U_S is an isometric linear operator on $L^2(\Sigma^+, \mu^+)$. For $x = 0, 1$ and $\omega = (\omega_1, \omega_2, \dots) \in \Sigma^+$, we denote by $x * \omega \in \Sigma^+$ the sequence $(x, \omega_1, \omega_2, \dots) \in \Sigma^+$.

We remark that the one-sided full 2-shift space Σ^+ is also regarded as a compact topological group. Actually, Σ^+ is a compact group defined by the product of the groups $\mathbb{Z}/2\mathbb{Z}$'s. Then the normalized Haar measure on this group is identical with the measure μ^+ defined above. For $k \in \mathbb{N}$, we define a character on Σ^+, which is a continuous group homomorphism from Σ^+ to $\mathbb{C}$, by

$$\rho_k : \Sigma^+ \longrightarrow \{1, -1\}; \omega = (\omega_1, \omega_2, \dots) \mapsto (-1)^{\omega_k}.$$

For $n \in \mathbb{Z}_{\geq 0}$, let $n = \sum_{i=0}^{\infty} s_i 2^i$ be the 2-adic expansion. Then we define the *n-th Walsh function* W_n by $W_n := \prod_{k=0}^{\infty} \rho_{k+1}^{s_k}$. The set of Walsh functions $\{W_n\}_{n=0}^{\infty}$ constitutes the whole characters of Σ^+, thus, by the representation theory of compact topological groups, $\{W_n\}_{n=0}^{\infty}$ is an orhothonomal basis of $L^2(\Sigma^+, \mu^+)$.

First, we give several basic properties of U_S and V_S on $L^2(\Sigma^+, \mu^+)$.

Proposition 2.1. 1. For $f \in L^2(\Sigma^+, \mu^+)$, we have

$$(V_S f)(\omega) = 2^{-1} f(0 * \omega) + 2^{-1} f(1 * \omega). \tag{2.1}$$

2. For $n \in \mathbb{Z}_{\geq 0}$, we have

$$U_S W_n = W_{2n} \tag{2.2}$$

$$V_S W_n = \begin{cases} W_{n/2} & \text{if } n \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

3. The spectrum of V_S is described as follows:

$$\text{point spectrum: } \{z \in \mathbb{C} : |z| < 1\} \cup \{1\},$$
$$\text{continuous spectrum: } \{z \in \mathbb{C} : |z| = 1, z \neq 1\}.$$

   Moreover, unless $z = 1$, the eigenspaces of the point spectrums z are infinite dimensional.

We define the function $h : \Sigma^+ \to [0, 1]$ by

$$h(\omega) = \sum_{i=1}^{\infty} \omega_i 2^{-i} \tag{2.4}$$

We note that h is a continuous and sujective map, and induces a homeomorphism outside the countable subset of periodic sequences, namely, h is a homeomorphism from $\Sigma^+ \setminus \{\omega = (\omega_i)_i : \omega_i = \omega_{i+r} \text{ for some } r\}$ onto $[0, 1] \setminus \mathbb{Z}[2^{-1}]$, where $Z[2^-1] = \{n2^{-m} : m, n \in \mathbb{Z}\}$. In particular, the pullback h^* induces a isomorphism between the Hilbert spaces:

$$h^* : L^2([0, 1], \mathrm{d}x) \cong L^2(\Sigma^+, \mu^+); x \mapsto h. \tag{2.5}$$

We remark that this isomorphism gives compatibility between Perron-Frobenius operators associated to the shift map and the 2-adic Rényi map. More precisely, let T be the 2-adic Rényi map, which is a map on $[0, 1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ 2x - 1 & \text{if } x \in (1/2, 1]. \end{cases} \tag{2.6}$$

Then, we have

$$h^* V_T = V_S h^*. \tag{2.7}$$

As in the following proposition, those of variable h is also polynomials of variable h of the same degree under the action of V_S:

Proposition 2.2. We have

$$h(0 * \omega) = 2^{-1} h(\omega)$$
$$h(1 * \omega) = 2^{-1} + 2^{-1} h(\omega).$$

In particular,

$$V_S[h^n] = 2^{-n} h^n + q(h),$$

where $q(h) \in \mathbb{C}[h]$ is a polynomial of variable h of degree smaller than n.

Computation of generalized spectrum We compute the generalized spectrum for V_S. First, we specify the test space X to determine a rigged Hilbert space. Let $X_n := \sum_{i=0}^{n} \mathbb{C} h^i$ be a $\mathbb{C}$-linear subspace of dimension $n+1$ in $L^2(\Sigma^+, \mu^+)$. We equip X_n with the usual topology of $\mathbb{C}^{n+1}$. Then we define the space

$$X := \varinjlim_{n} X_n = \mathbb{C}[h],$$

equipped with the inductive limit topology.

Proposition 2.3. The space X is a dense subspace of $L^2(\Sigma^+, \mu^+)$ and has a stronger topology than $L^2(\Sigma^+, \mu^+)$.

We note that X is a Montel space. Hence, the rigged Hilbert space $X \subset L^2(\Sigma^+, \mu^+) \subset X'$ is well-defined. Then X and X_n satisfy the following proposition:

Proposition 2.4. For $n \in \mathbb{Z}_{\geq 0}$, we have

$$V_S X_n \subset X_n.$$

Moreover, the representation matrix of $V_S|_{X_n}$ associated to the basis $1, h, \ldots, h^n$ is a upper triangular matrix whose diagonal components are $1, 2^{-1}, \ldots, 2^{-n}$. In particular, V_S induces a continuous linear operator on X.

Corollary 2.5. For each $n \in \mathbb{Z}_{\geq 0}$, the eigenspace of V_S in X corresponding to 2^{-n} is 1-dimensional.

We denote by $\Phi_n \in X_n$ the unique eigenvector of V_S corresponding to 2^{-n} that is a monic polynomial of variable h of degree n. We define the dual basis $\{\Phi'_n\}_{n=0}^{\infty} \subset X'$ defined as continuous linear functionals on X such that $\langle \Phi'_i | \Phi_j \rangle = \delta_{i,j}$.

Theorem 2.6. The generalized spectrum of the Perron-Frobenius operator V_S with respect to the triplet $X \subset L^2(\Sigma^+, \mu^+) \subset X'$ is given by $\{2^{-n}\}_{n \geq 0}$. As a linear operator on X', V_S has a spectral decomposition in the following sense:

$$V_S = \sum_{n=0}^{\infty} 2^{-n} |\Phi_n\rangle\langle\Phi'_n|.$$

Here the equality is in the sense of the point-wise convergence topology (weak * topology) of X'.

Proof. Let $R_\lambda = (\lambda - V_S)^{-1}$ be the resolvent operator of V_S. For $f, g \in L^2(\Sigma^+, \mu^+)$, we have Then we have

$$\begin{aligned} R_\lambda[f][g] &= \left((\lambda - V_S)^{-1} f, g\right) \\ &= \frac{1}{\lambda}\left((1 - V_S/\lambda)^{-1} f, g\right). \end{aligned}$$

When $|\lambda| > 1$, $\|V_S/\lambda\| < 1$, and the Neumann series is applied to yield

$$R_\lambda[f][g] = \frac{1}{z} \sum_{r=0}^{\infty} \frac{1}{\lambda^r} \left(V_S^r f, g\right).$$

Let us construct an analytic continuation of R_λ as an operator from X to X'. For this poupose, suppose $f, g \in X$. Let $f = \sum_{i=0}^{m} c_i \Phi_i$. Then, Corollary 2.5 shows

$$\begin{aligned} R_\lambda[f][g] &= \langle R_\lambda f | g\rangle \\ &= \frac{1}{\lambda} \sum_{r=0}^{\infty} \sum_{i=0}^{m} \frac{c_i}{(2^i \lambda)^r} \langle\Phi_i | g\rangle \\ &= \sum_{i=0}^{m} \frac{c_i \langle\Phi_i | g\rangle}{\lambda - 2^{-i}}. \end{aligned}$$

The last term gives the analytic continuation of $R_\lambda[f]$ from the region $|\lambda| > 1$ to $|\lambda| < 1$ as an X'-valued function for each $f \in X$, and R_λ is a continuous linear operator from X to X'. As for the spectral decomposition of V_S, for

$f = \sum_{i=0}^{m} c_i \Phi_i \in X$, we have

$$\begin{aligned} V_S f &= \sum_{i=0}^{m} c_i 2^{-i} \Phi_i \\ &= \sum_{i=0}^{m} 2^{-i} \langle \Phi_i' | f \rangle \Phi_i \\ &= \left[\sum_{i=0}^{\infty} 2^{-i} |\Phi_i\rangle \langle \Phi_i'| \right] f. \end{aligned}$$

□

Remark 2.7. Antoniou and Tasaki computed the generalized spectrum for the Rényi map in [1]. They prove that the generalized spectrum of Perron-Frobenius operator associated to 2-adic Rényi map is $1, 2^{-1}, 2^{-2}, \ldots,$ and the corresponding eigenvectors are *Bernoulli polynomials* $\{B_n(x)\}_{n=0}^{\infty}$, which are defined as coefficients of the following Taylor expansion:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

Theorem 2.6 is compatible with their result through the isomorphism h^* of (2.5). Namely, the Bernoulli polynomial $h^* B_n$ is identical with Φ_n. In fact, the Bernoulli polynomial satisfies (a special case of) Raabe's multiplication theorem[4]:

$$\frac{1}{2} B_n(x) + \frac{1}{2} B_n(x + 1/2) = 2^{-n} B(2x).$$

We can prove that the left hand side coincides with $(V_T B_n)(2x)$, and thus, Raabe's multiplication theorem means that Bernoulli polynomials B_n's are eigenvectors of V_T correspond to 2^{-n}'s, respectively. Since the test space X in $L^2(\Sigma^+, \mu^+)$ corresponds to the space of polynomials in $L^2([0,1], \mathrm{d}x)$ via h^*, by the same argument of the proof of Proposition 2.4, B_n is characterized as a unique eigenvector with respect to 2^{-n} in the space of polynomials, and thus $h^* B_n = \Phi_n$.

Remark 2.8. It might be natural to consider X as a linear space generated by Walsh functions $\{W_k\}_{k=0}^{\infty}$, or, equivalently, X as the space of locally constant functions on Σ^+ (they sometimes referred to as the space of smooth functions on Σ^+). However, we never have an analytic continuation of R_λ to the domain $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. In fact, suppose we have an analytic continuation R_λ to $\lambda = w \in \mathbb{C}$ with $|w| < 1$. Let $a \in \mathbb{R}$ be a real number such that $|w| < a < 1$. We define $\phi_m := a^m W_{2^m}$, and let $h_m(\lambda) := R_\lambda[\phi_m][W_1]$ be a holomorphic function defined on a connected open subset V containing w and $\{z \in \mathbb{C} : |z| > 1\}$. Since R_λ is continuous linear operator from X to X' and $\phi_m \to 0$ in X, we have $h_m(w) \to 0$. On the other hand, for $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, we have

$$\begin{aligned} h_m(\lambda) &= \frac{1}{\lambda} \sum_{r=0}^{\infty} \frac{1}{\lambda^r} \langle a^m W_{2^{m-r}} | W_1 \rangle \\ &= \frac{a^m}{\lambda^{m+1}}. \end{aligned}$$

Thus we conclude that h_m is a holomorphic function on V such that $h_m(\lambda) = a^m/\lambda^{m+1}$ if $|\lambda| > 1$. By the identity theorem, $h_m(w) = a^m/w^{m+1}$, and we have $|h_m(w)| \to \infty$ as $m \to \infty$, which is contradiction.

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