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Fiber Topology Meets Applications 2

Editors :

Daisuke Sakurai
Osamu Saeki
Shigeo Takahashi
Hamish Carr
Takahiro Yamamoto
Naoki Hamada

Institute of Mathematics for Industry
Kyushu University

About the Mathematics for Industry Research

The Mathematics for Industry Research was founded on the occasion of the certification of the Institute of Mathematics for Industry (IMI), established in April 2011, as a MEXT Joint Usage/Research Center – the Joint Research Center for Advanced and Fundamental Mathematics for Industry – by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) in April 2013. This series publishes mainly proceedings of workshops and conferences on Mathematics for Industry (MfI). Each volume includes surveys and reviews of MfI from new viewpoints as well as up-to-date research studies to support the development of MfI.

October 2018
Osamu Saeki
Director
Institute of Mathematics for Industry

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Hamish Carr
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Naoki Hamada

Daisuke Sakurai
Kyushu University
February, 2022

Introduction

In differential topology, the study of the topology of fibers, known as fiber topology, extends the Morse theory, of functions, to that of maps. This establishes a thorough exploration of topological transitions of inverse images. The topological analysis of the level set, from which fiber topology-based data analysis originates, has been proven powerful for understanding today's data whose size and complexity are overwhelming. We have worked on establishing data analysis utilizing fiber topology with regard to theory and computation, and now are on the stage to co-design applications and further developments for them.

Continuing the effort in the previous meeting, we propose to gather researchers from academia and the industry to discuss the future directions. The participants included mathematics, computation, and applications. For this, participants bring themes from broad spectra of practitioners including those from mathematics, topological analysis in visualization, the game industry and environmental sciences.

Our collaboration for application in the game industry started by designing and automating game music developments for mobile devices. We have since expanded the aim to into general optimization problems for mobile games. A key problem is to understand multi-objective optimization, for which fiber topology can contribute by designing and understanding benchmarks. Another application is environmental science, for which we discuss integrating fiber topology into HPC simulations.

All these pose a challenge to state-of-the-art algorithms. The current, rather loose, mathematics found in the computer science literature meets a more rigorous treatment by mathematicians and we will discuss the direction we can take.

Acknowledgement

This forum was co-sponsored by Kyushu University *Institute of Mathematics for Industry* under FY2022 IMI Joint Usage/Research Program and *Pan-Omics Data-Driven Research Innovation Center* of the same university.

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九州大学IMI共同利用・一般研究-研究集会(II)　公開プログラム

Fiber Topology Meets Applications 2

日　　時：2021年12月1日（水）17：00 ～ 18：15
2021年12月2日（木）17：00 ～ 18：15
2021年12月3日（金）17：00 ～ 18：40
場　　所：九州大学 伊都キャンパス ウエスト1号館 D棟4階
IMIコンファレンスルーム（W1-D-414）
（対面＋オンラインのハイブリッド形式）
研究代表者：櫻井 大督（九州大学情報基盤研究開発センター附属
汎オミクス計測・計算科学センター）

※プログラムは都合により変更になる場合がありますので予めご了承ください。
最新情報はホームページをご覧ください。

＜プログラム＞

12月1日（水）

17:00-17:25
Fiber Topology Meets Applications: This Year
Daisuke Sakurai (Kyushu University)

17:25-17:50
Fibers, Covers, Tessellation & Partial Orders: Different Approaches to Analysing Multi-Variate Data
Hamish Carr (University of Leeds)

17:50-18:15
On-line Diagnostics in climate simulations – Status and perspectives
Bastian Kern and Hiroshi Yamashita (German Aerospace Center)

12月2日（木）

17:00-17:25
Explicitly Multimodal Benchmarks for Multi-Objective Optimizations
Reiya Hagiwara (Kyushu University), Ryosuke Ota (Kyushu University), Naoki Hamada (KLab Inc.), Takahiro Yamamoto (Tokyo Gakugei University), Daisuke Sakurai (Kyushu University)

17:25-17:50
The Centroids of Isosurface Components for All Isovalues
Akito Fujii, Daisuke Sakurai and Kenji Ono (Kyushu University)

17:50-18:15
Thinking about multi-objective optimization problems topologically
Likun Liu, Daisuke Sakurai and Kenji Ono (Kyushu University)

12月3日（金）

17:00-17:25
Topological invariants of smooth map germs of manifolds with boundary
Takahiro Yamamoto (Tokyo Gakugei University)

17:25-17:50
Reeb diagram and visualization of monodromy
Osamu Saeki (Institute of Mathematics for Industry, Kyushu University)

17:50-18:15
Combinatorics Meets Topology at Piecewise Linear Reeb Spaces
Peter Hristov (University of Leeds)

18:15-18:40
Concluding Remarks
Daisuke Sakurai (Kyushu University)

Fiber Topology Meets Applications: This Year

Daisuke Sakurai
2021-12-01

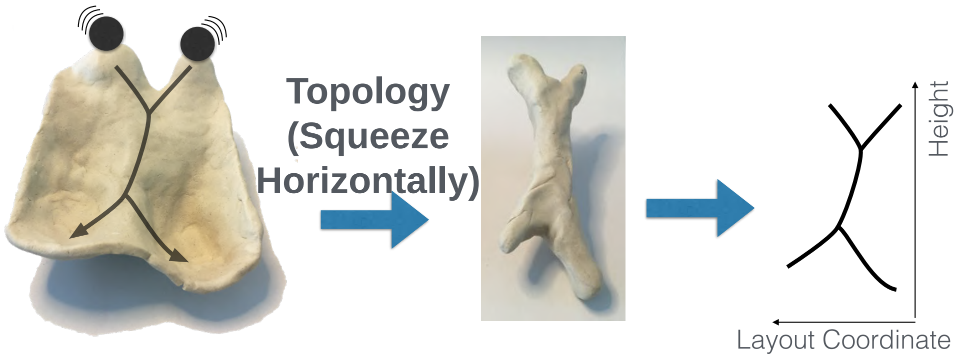
This Forum

- Forum for fiber topology & Reeb spaces (Generalized contour topology & Reeb graphs)
- From math, compsci & applications
- Share recent progress in retreat-style

My Talk (25min Total)

1. Introduction to the forum
2. Progress on our Reeb space computation

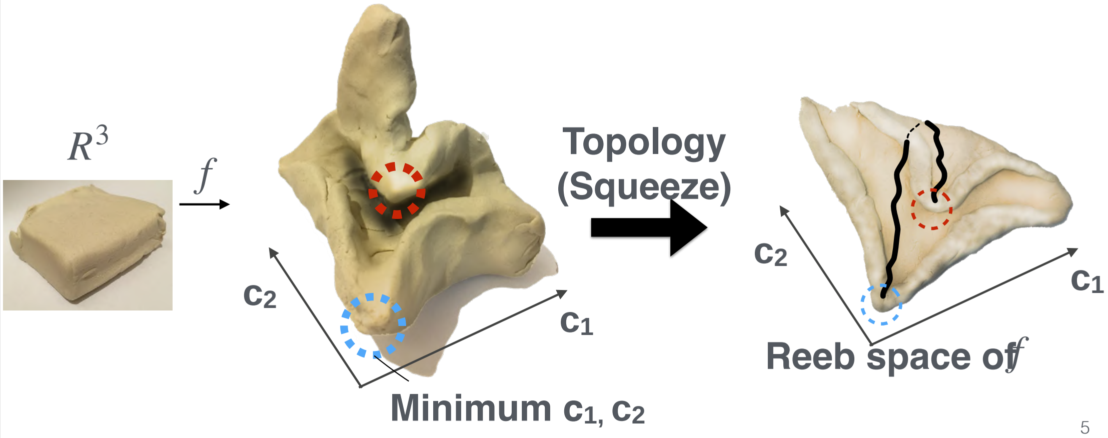
Reeb Graph: *Compactifies* Landscape



$f : R^2 \rightarrow$ **Height**

Contour Tree

Reeb Space: Compactifies Multi-Value *Landscape*



Talks from Target Applications

- **Machine learning**
  - Hyper-parameter tuning **(Likun on Thu)**
- **Multi-objective optimization**
  - Benchmarking **(Reiya on Thu)**
- **Atmospheric sciences**
  - HPC simulations **(Bastian & Hiroshi today)**
  - Multi-objective optimization **(Dec 15 @ IMI)**

Talks from Topological Analysis

- Review of related data analyses (Hamish today)
  - JCN, Mapper, etc.
- Classification of Reeb spaces for PL maps (Petar on Fri)
- Centroids of isosurfaces (Akito on Thu)

Talks from Math

- Monodromy (twists of fibers)
  (Saeki-sensei on Fri)
- A study of topological invariants with visualization
  (Takahiro on Fri)

On-line Diagnostics in Climate Simulations - Status and Perspectives

Bastian Kern and Hiroshi Yamashita
Institute of Atmospheric Physics – Earth System Modelling
German Aerospace Center (DLR)
Oberpfaffenhofen

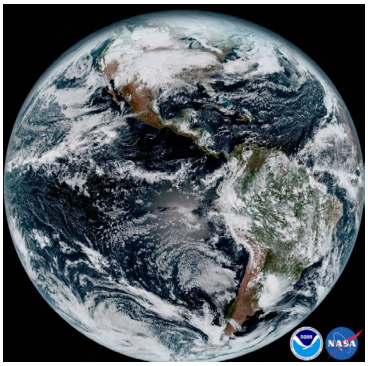
Overview

- Introduction
  - Climate modelling
  - Challenges of high data volume
- On-line Diagnostics (examples)
  - Exploiting the native time-step of the simulation
  - Get results, that are not possible off-line
  - In-situ analysis
- Perspectives
  - Upcoming project at DLR
- Summary / Discussion

Introduction

(Numerical) Modelling in a nutshell

- There is no second Earth (to experiment with)



NOAA/NASA
https://www.nasa.gov/image-feature/new-weather-satellite-sends-first-images-of-earth

Wikimedia, user Hellerick
https://commons.wikimedia.org/wiki/File:Division_of_the_Earth_into_Gauss-Krueger_zones_-_Globe.svg

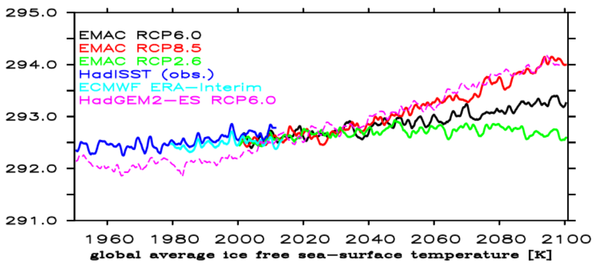
(Numerical) Modelling in a nutshell

- There is no second Earth (to experiment with)
- A mathematical model
  → based on physical equations
  → coupled system of (non-linear) partial differential equations
  → solved numerically on a “supercomputer”

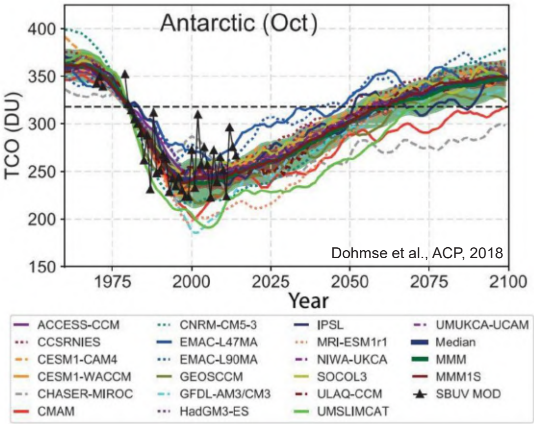
https://www.dkrz.de/about/media/galerie/Media-DKRZ/hlre-3

Climate projections with EMAC

- Model simulations with 2 PByte output
- Contribution to Chemistry Climate Initiative (CCMI)
- Data for WMO Ozone Assessment and Intergovernmental Panel on Climate Change (IPCC)



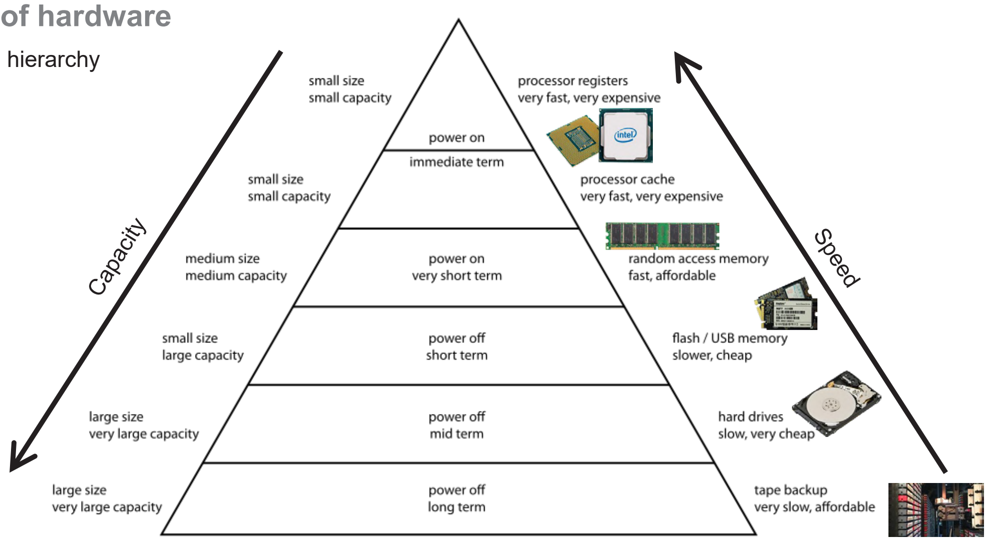
Total ozone column (DU), Antarctic, October



Dohmse et al., ACP, 2018

Limits of hardware

- Storage hierarchy



Capacity

Speed

small size
small capacity

power on
immediate term

processor registers
very fast, very expensive

small size
small capacity

processor cache
very fast, very expensive

medium size
medium capacity

power on
very short term

random access memory
fast, affordable

small size
large capacity

power off
short term

flash / USB memory
slower, cheap

large size
very large capacity

power off
mid term

hard drives
slow, very cheap

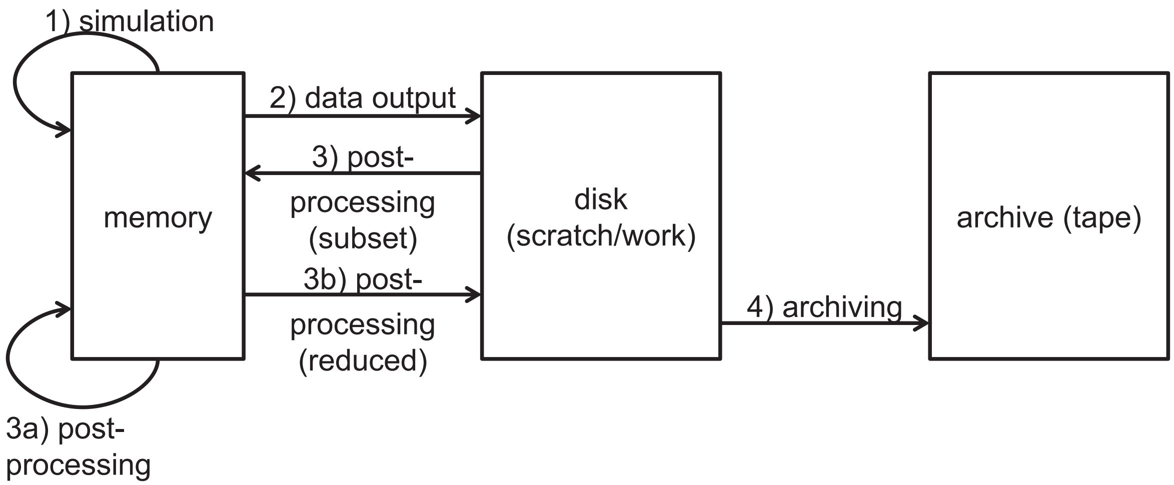
large size
very large capacity

power off
long term

tape backup
very slow, affordable

https://en.wikipedia.org/wiki/Memory_hierarchy

How does the I/O gap impact a typical workflow?



1) simulation

memory

2) data output

3) post-processing (subset)

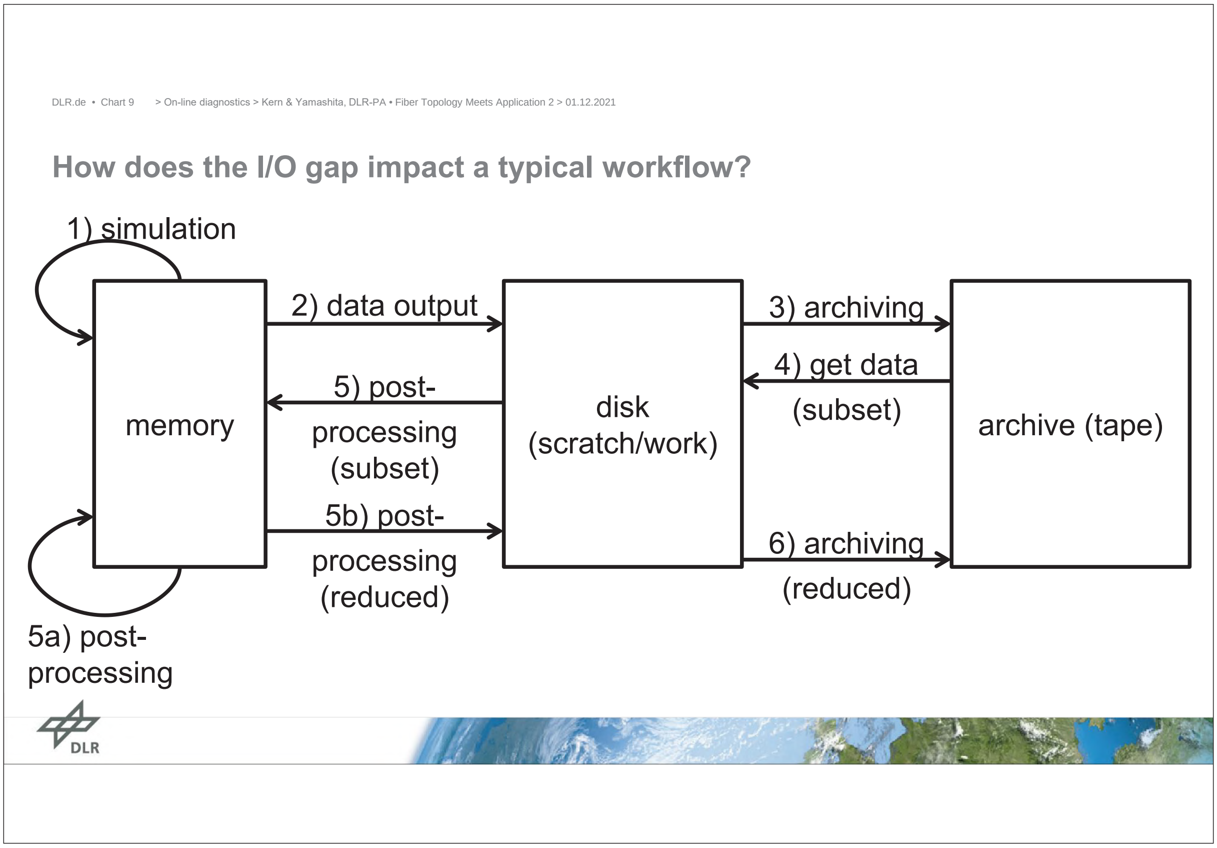
3b) post-processing (reduced)

disk (scratch/work)

4) archiving

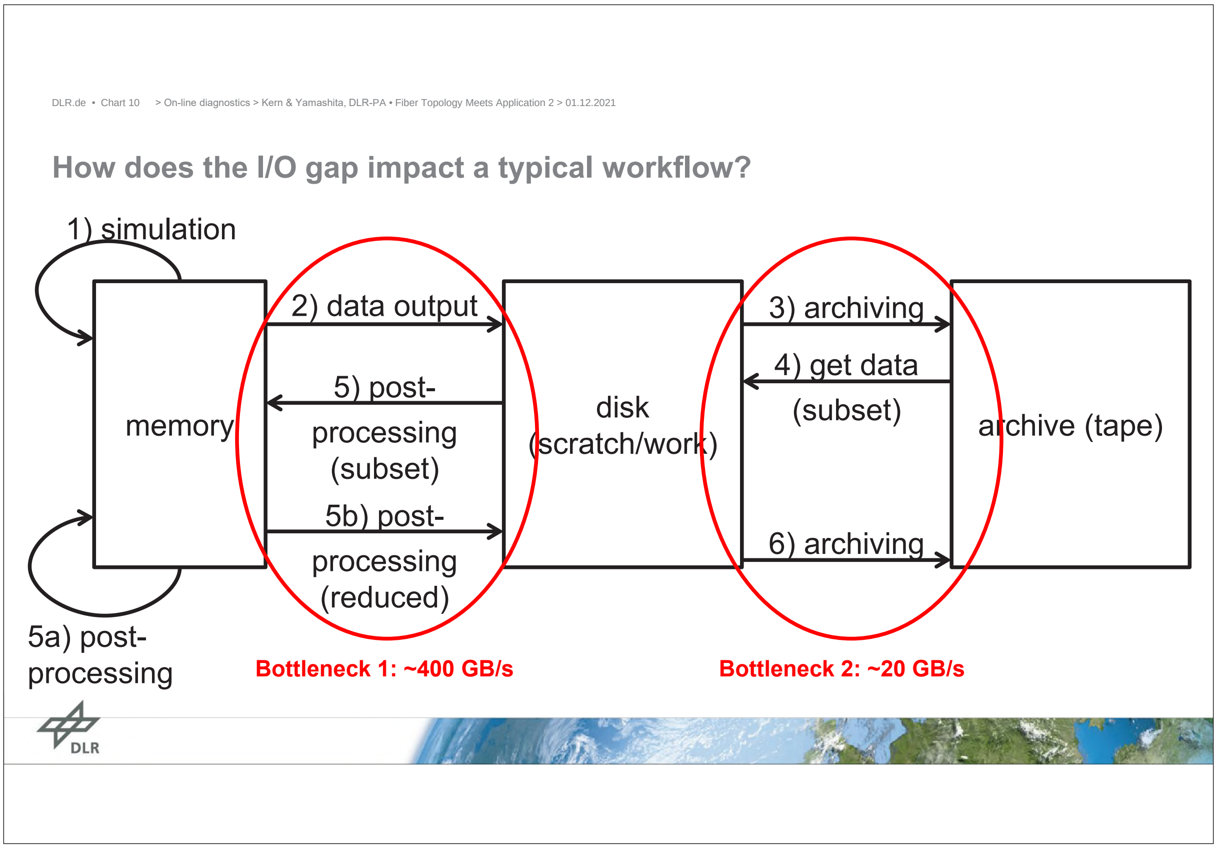
archive (tape)

3a) post-processing



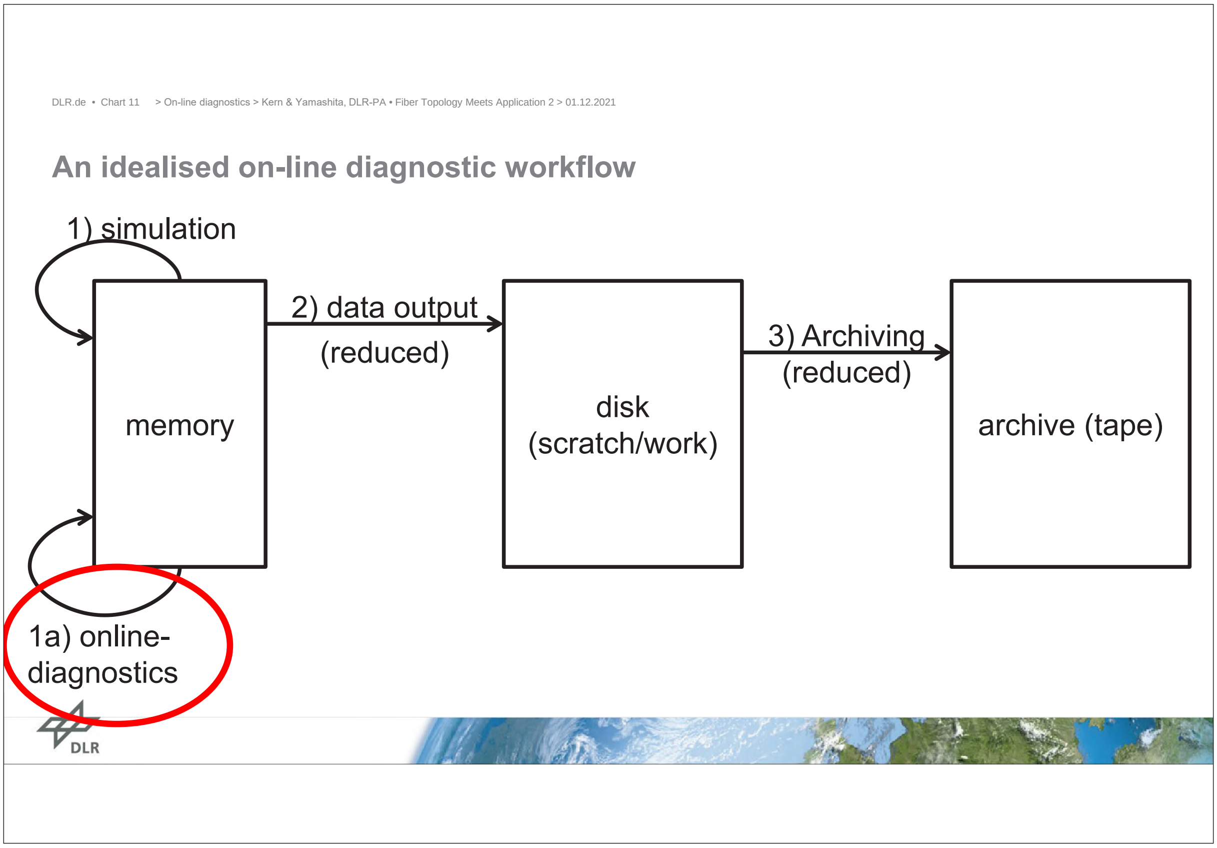
DLR.de • Chart 9 > On-line diagnostics > Kern & Yamashita, DLR-PA • Fiber Topology Meets Application 2 > 01.12.2021

How does the I/O gap impact a typical workflow?



DLR.de • Chart 10 > On-line diagnostics > Kern & Yamashita, DLR-PA • Fiber Topology Meets Application 2 > 01.12.2021

How does the I/O gap impact a typical workflow?



An idealised on-line diagnostic workflow

1) simulation

memory

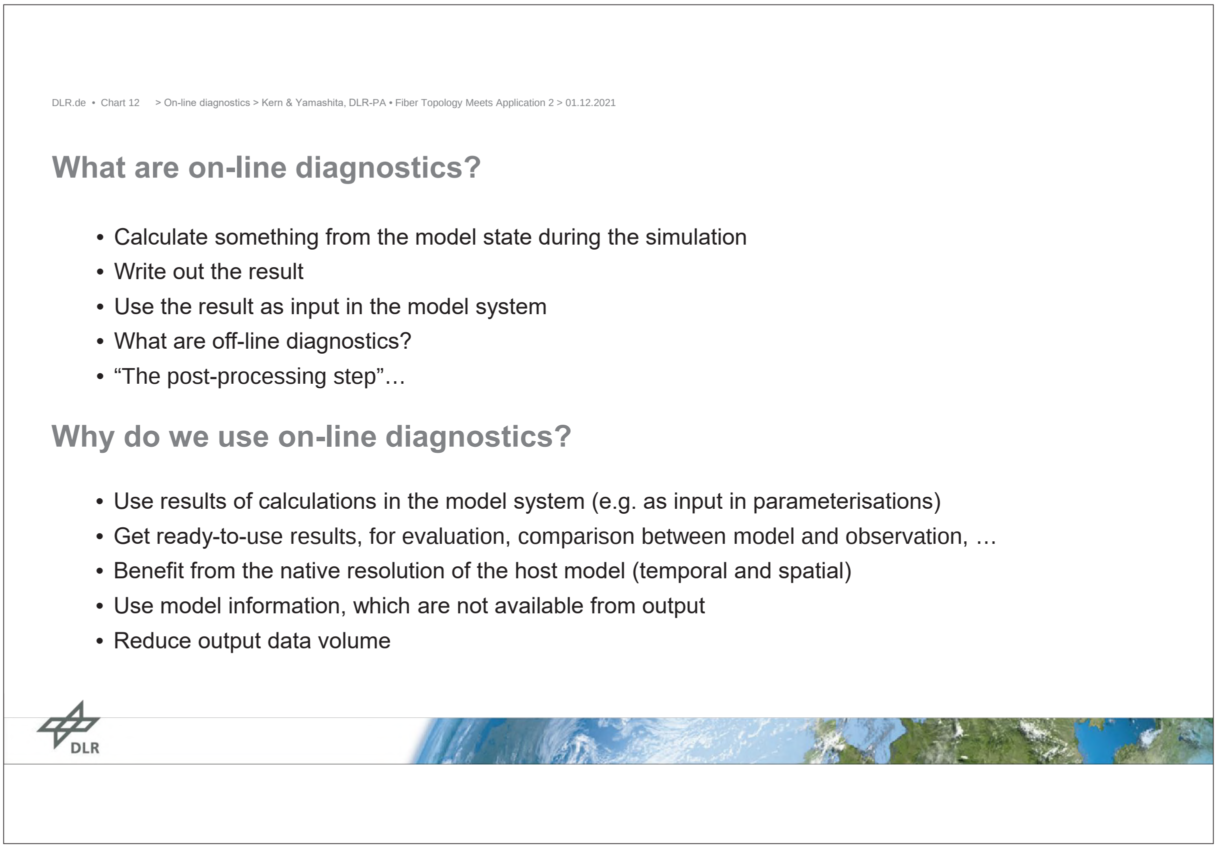
2) data output (reduced)

disk (scratch/work)

3) Archiving (reduced)

archive (tape)

1a) online-diagnostics



What are on-line diagnostics?

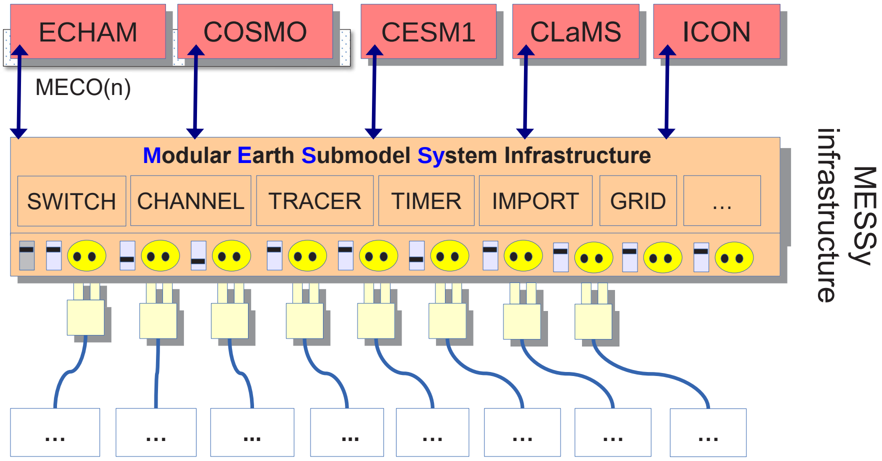
- Calculate something from the model state during the simulation
- Write out the result
- Use the result as input in the model system
- What are off-line diagnostics?
- “The post-processing step”…

Why do we use on-line diagnostics?

- Use results of calculations in the model system (e.g. as input in parameterisations)
- Get ready-to-use results, for evaluation, comparison between model and observation, …
- Benefit from the native resolution of the host model (temporal and spatial)
- Use model information, which are not available from output
- Reduce output data volume

Modular Earth Submodel System (MESSy)

EMAC = **E**CHAM/**M**ESSy **A**tmospheric **C**hemistry



https://www.messy-interface.org/

> 20 partner institutes

Framework to couple scientific codes to numerical weather prediction and climate models

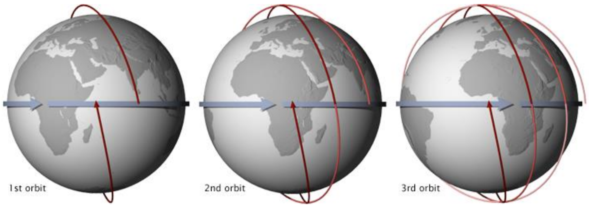
---

Make use of native temporal model resolution
Sampling along satellite swaths

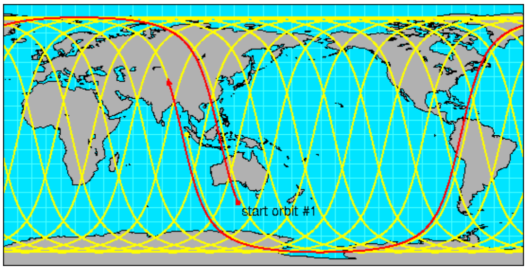
SORBIT - Sampling along sun-synchronous satellite ORBITs

Sun-synchronous orbit

- Equator crossing at approx. the same local time each day (and night)
- Consistent scientific observations
- Relatively constant angle between the Sun and the Earth's surface



Three consecutive orbits of a sun-synchronous satellite with an equatorial crossing time of 1:30 pm.



Ground paths of the multiple orbital revolutions during one day for a near-polar orbiting satellite.

https://en.wikipedia.org/wiki/Sun-synchronous_orbit
https://earthobservatory.nasa.gov/features/OrbitsCatalog/page2.php
http://tornado.sfsu.edu/geosciences/classes/m407_707/Monteverdi/Satellite/PolarOrbiter/Polar_Orbits.html

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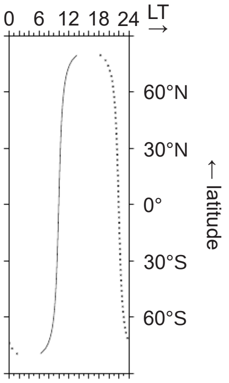
SORBIT

daily
SORBIT
output

5h model
output

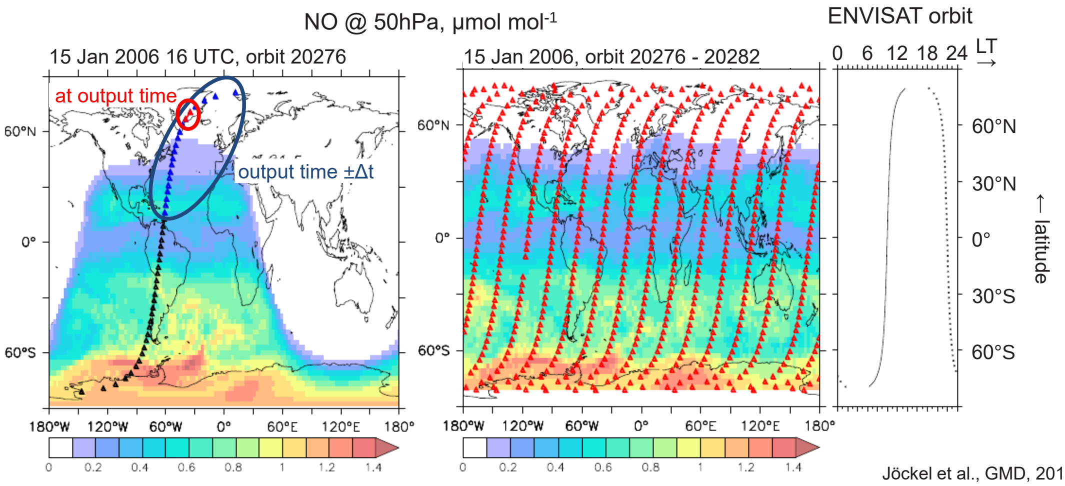
ENVISAT orbit



Jöckel et al., GMD, 2010
doi:10.5194/gmd-3-717-2010

SORBIT

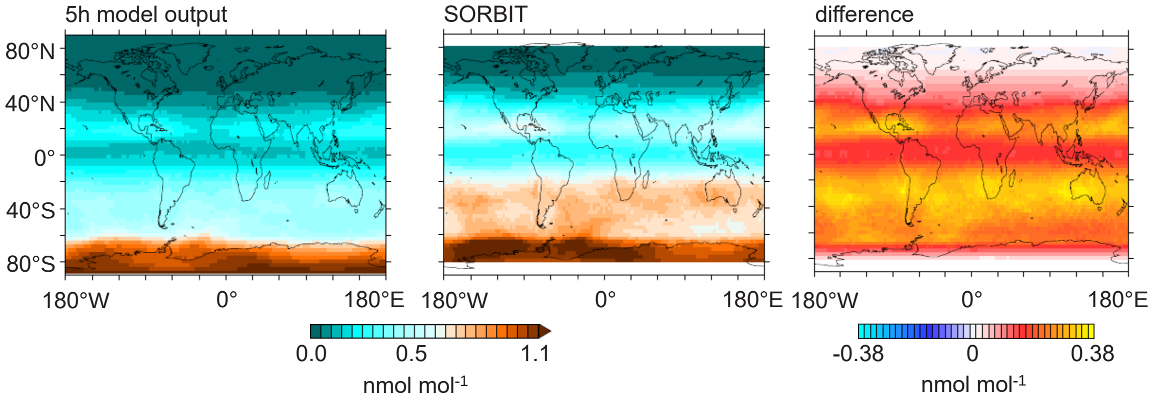
NO @ 50hPa, μmol mol^{-1}



Jöckel et al., GMD, 2010
doi:10.5194/gmd-3-717-2010

SORBIT

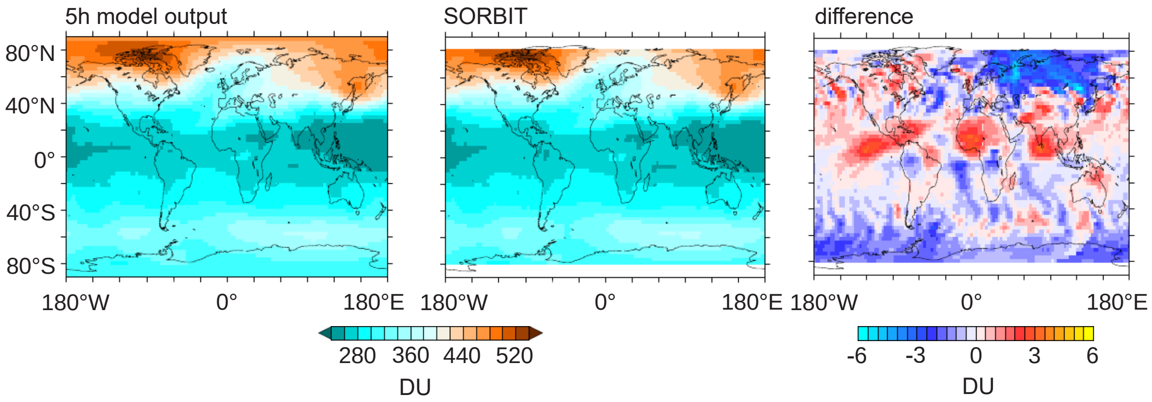
NO @ 50 hPa, monthly average (Jan 2006)



Jöckel et al., GMD, 2010
doi:10.5194/gmd-3-717-2010

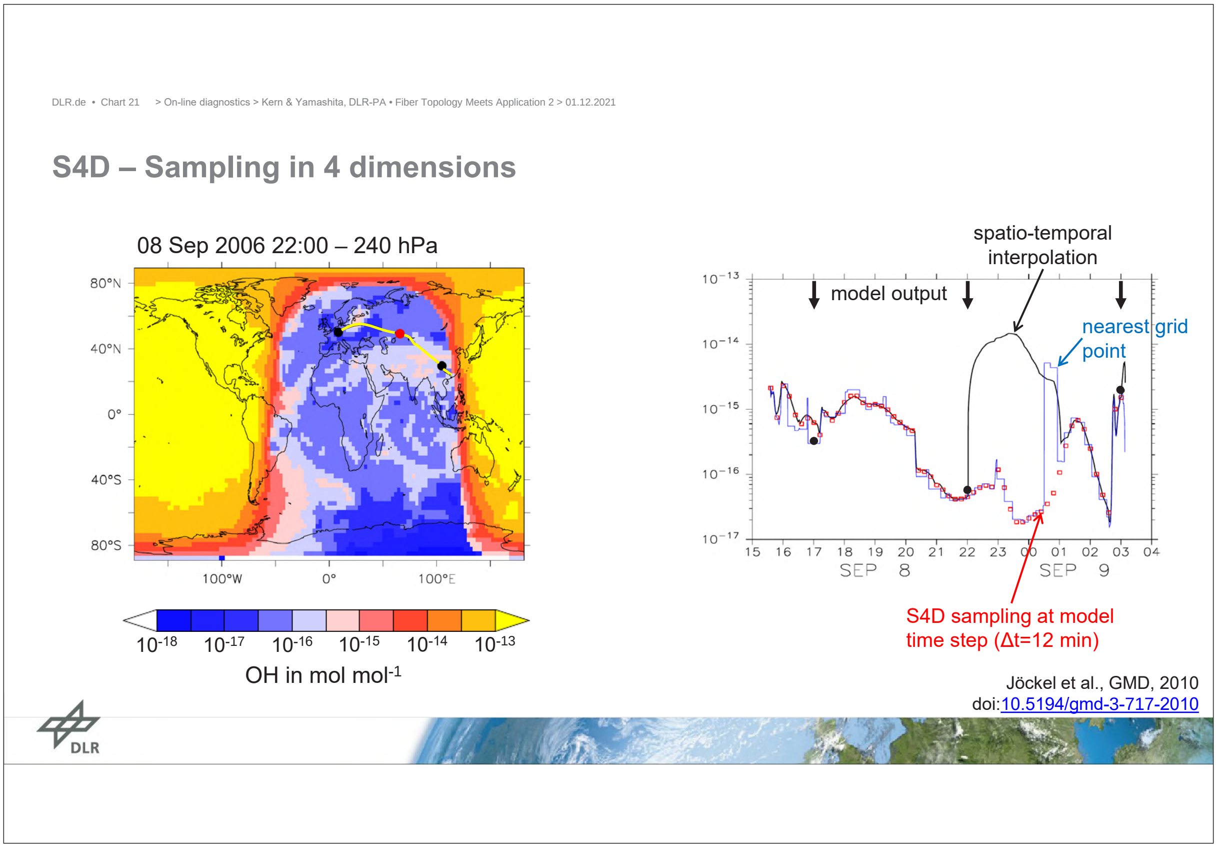
SORBIT

Total O_3 column, monthly average (Jan 2006)



Jöckel et al., GMD, 2010
doi:10.5194/gmd-3-717-2010

Make use of native temporal model resolution – 2
Sampling in space and time



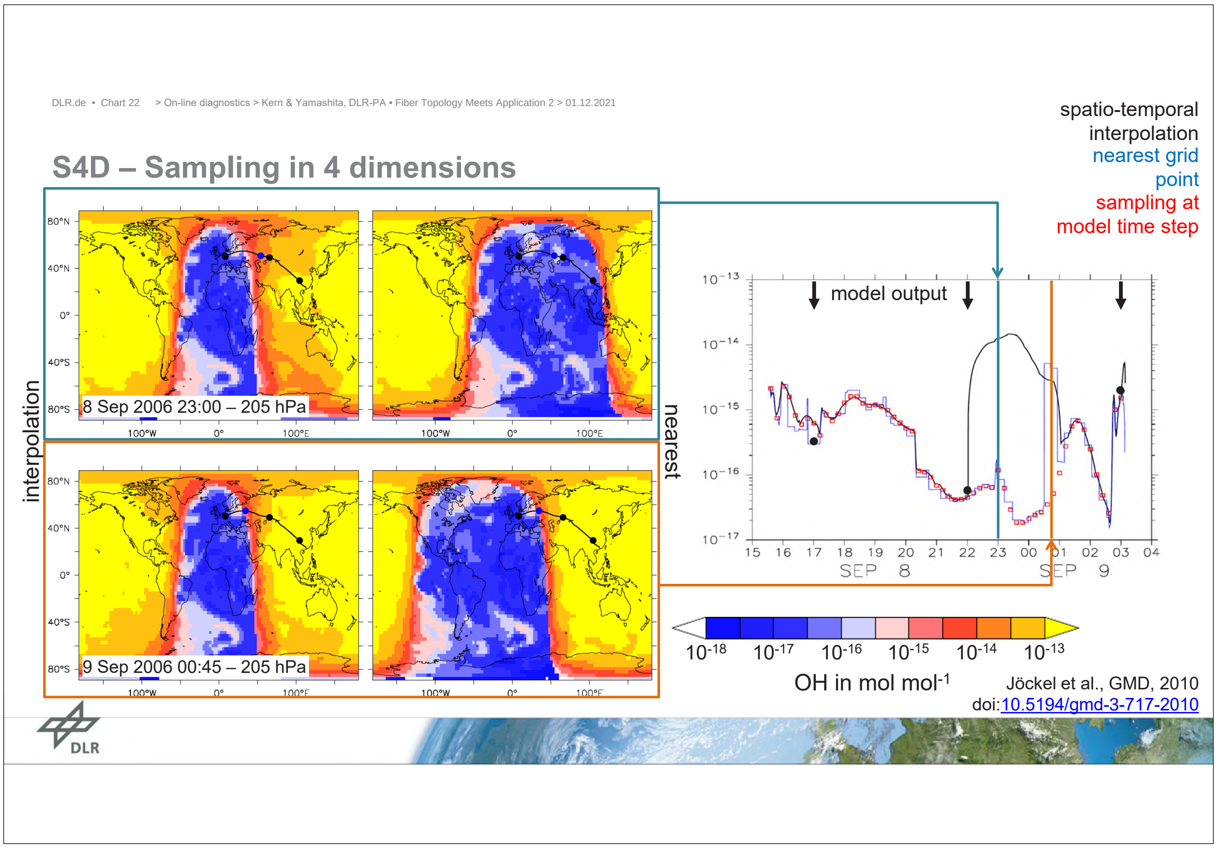
DLR.de • Chart 21 > On-line diagnostics > Kern & Yamashita, DLR-PA • Fiber Topology Meets Application 2 > 01.12.2021

S4D – Sampling in 4 dimensions

08 Sep 2006 22:00 – 240 hPa

Jöckel et al., GMD, 2010

doi:10.5194/gmd-3-717-2010



DLR.de • Chart 22 > On-line diagnostics > Kern & Yamashita, DLR-PA • Fiber Topology Meets Application 2 > 01.12.2021

S4D – Sampling in 4 dimensions

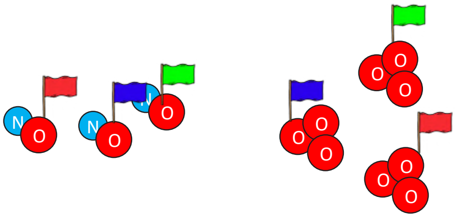
Jöckel et al., GMD, 2010

doi:10.5194/gmd-3-717-2010

Get results, that are not possible from off-line data
Tagging of chemical emissions

TAGGING

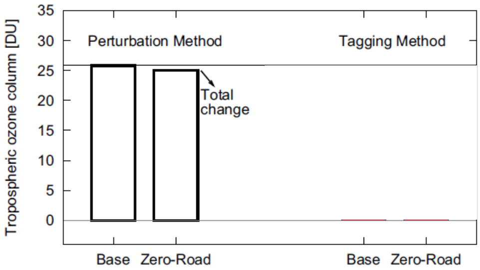
“Classic” - Differences	“reference run” minus “perturbed run”, “ sensitivity study”
“Tagging”	marking emissions and tracking

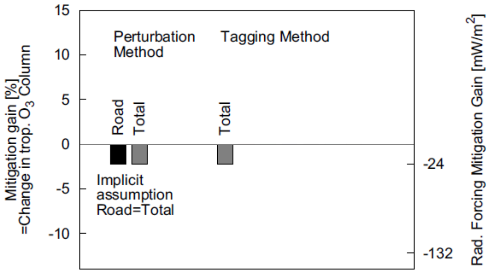


Grewe et al., GMD, 2017
doi:10.5194/gmd-10-2615-2017

TAGGING

Contribution of road traffic emissions to O_3



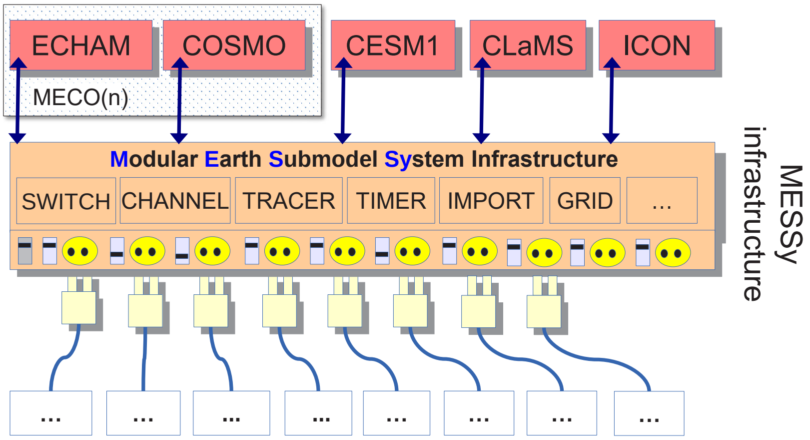


- Ozone decreases because ozone from road traffic decreases (12%)
- Ozone net production rates increase ⇒ ozone from other sectors increases (-10%)
- Net Ozone change = 2%

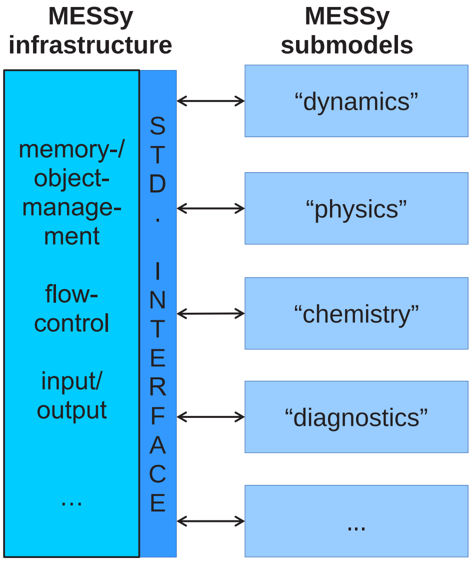
Current implementation in our model system

Interfaces

- Climate/weather model
- The Modular Earth Submodel System (MESSy)
- Submodels
- Implementation: Fortran



Jöckel et al., ACP, 2005, doi:10.5194/acp-5-433-2005
Jöckel et al., ACP, 2006, doi:10.5194/acp-6-5067-2006
Jöckel et al., GMD, 2010, doi:10.5194/gmd-3-717-2010



“communication” between individual components through mutual access of distributed objects via standard interfaces

expandability without intervention in other components

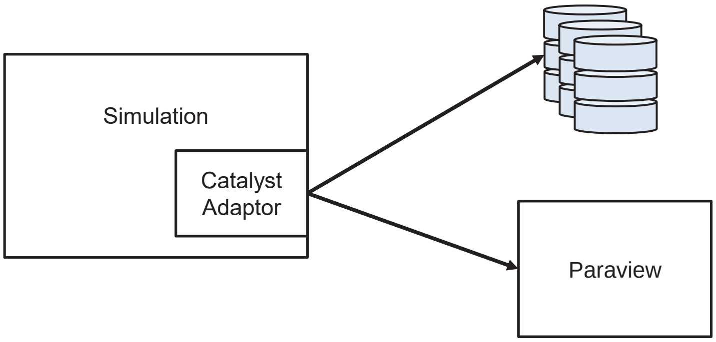
- scalable development
- coexistence of alternatives

→ **“community”-ansatz**

Monitor and analyse simulations in-situ

Online Analysis and Visualisation

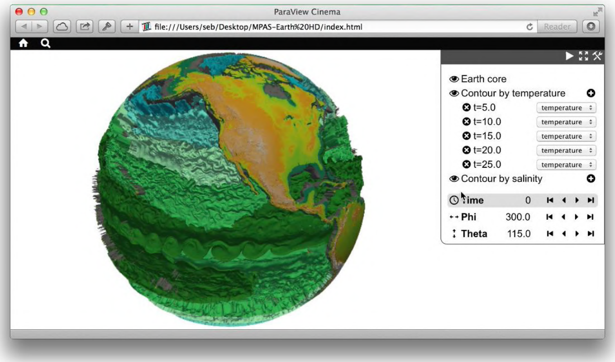
- Paraview / Catalyst



https://www.paraview.org/in-situ

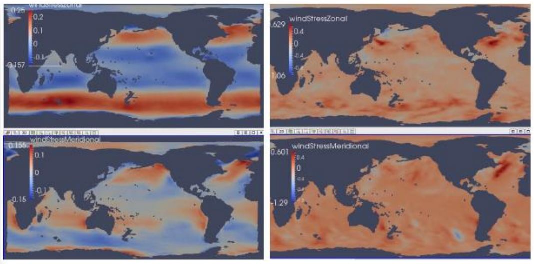
Online Analysis and Visualisation

- Catalyst Adaptor for MPAS Ocean



Time 0
Phi 300.0
Theta 115.0

In-situ visualisation



diagnosing model setup errors

Ayachit et al., ISAV2015, 2015
doi:10.1145/2828612.2828624
https://www.paraview.org/in-situ

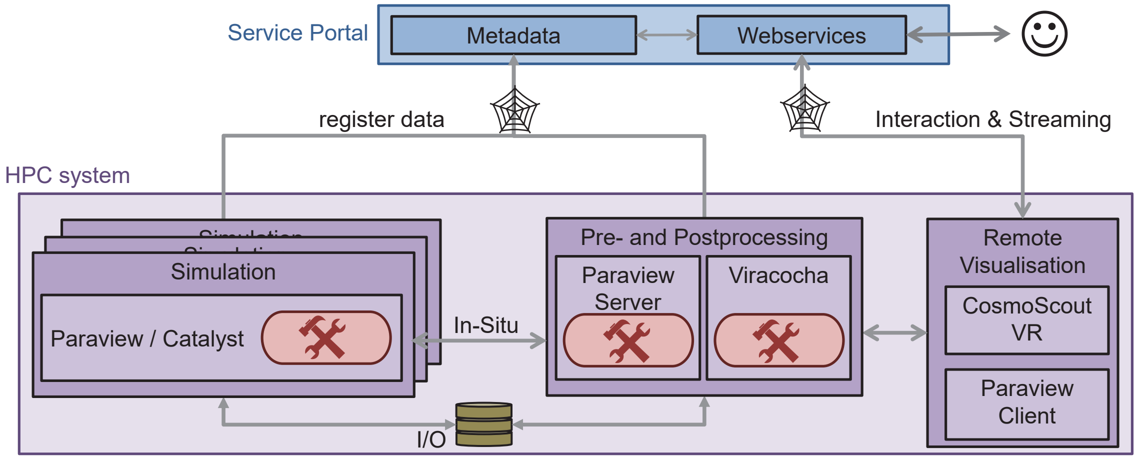
Perspectives

DLR Project “VisPlore” (VISual & exPLORE)

- 6 DLR Institutes
- 3 years, starts 01/2022

- Standardised interface in simulation codes of different disciplines (CFD, climate simulation, EO, …)
- Provide metadata and data fields
- Paraview/Catalyst
- Interactive exploration
- Python-based (inter-disciplinary) toolset
- Web-based User-portal

DLR Project “VisPlore” (VISual & exPLORE)



adapted from VisPlore project plan

Summary

- Need for On-line diagnostics and examples
- Current Implementation via (Fortran) “Submodels”
- Future plan of common interface for different disciplines
- Common tool-set of algorithms (Python)

- Discussion:
  - Will such an interface support implementation of novel algorithms?
  - Will it enhance opportunity to apply new tools from other disciplines to gain new insights?
  - Whole history stored off-line vs. high-frequent, but limited number of timesteps on-line?

Explicitly Multimodal Benchmarks for Multi-Objective Optimizations

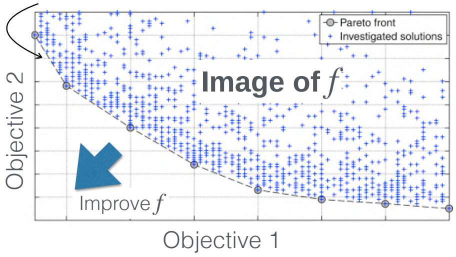
**Reiya Hagiwara (Kyushu University), Ryosuke Ota (Kyushu University),
Naoki Hamada (KLab Inc.), Takahiro Yamamoto (Tokyo Gakugei University),
Daisuke Sakurai (Kyushu University)**

2021-11-29

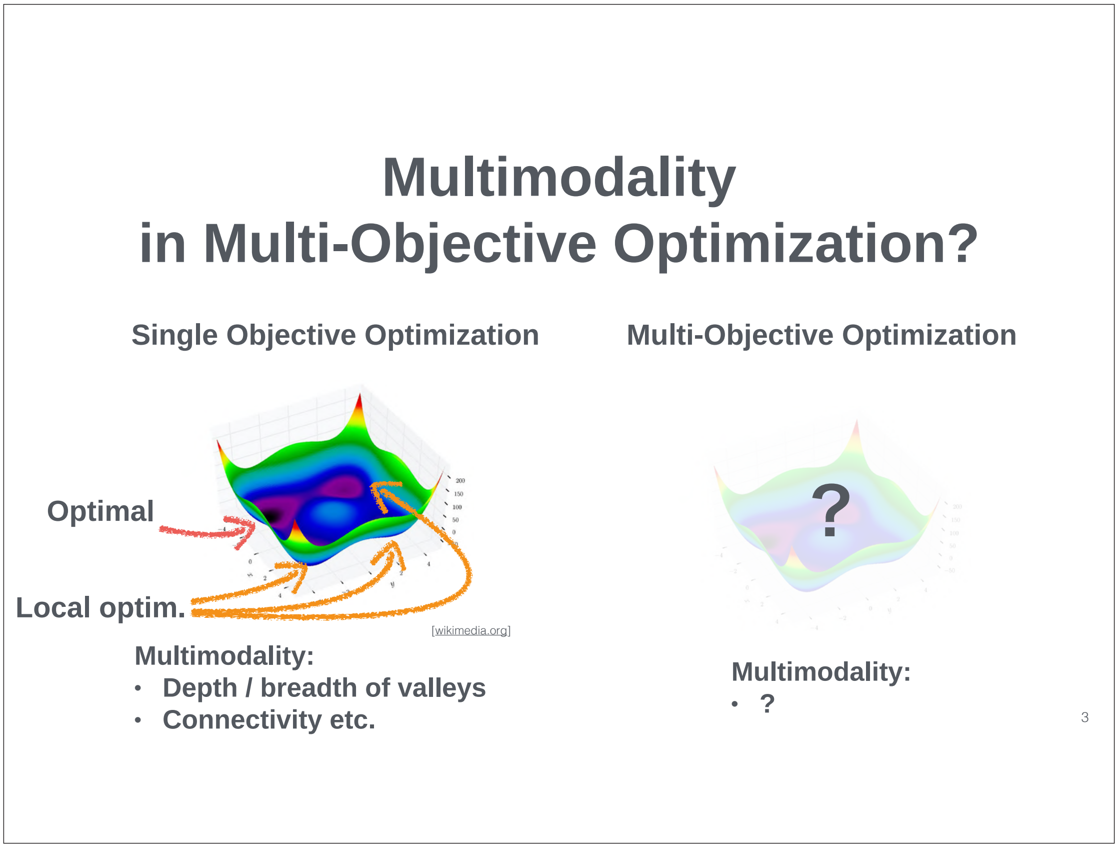
Multi-Objective Optimization

- **Multiple objectives $f : \mathbb{R}^n \to \mathbb{R}^m$**
- **An "optimal" x:**
  - **Pareto: if $f_i(x) > f_i(x')$ then $f_j(x) < f_j(x')$**
  - **Locally Pareto**
    - **Pareto optim. in neighborhood of x**

"Optimal": Pareto front



[https://www.sciencedirect.com/topics/engineering/pareto-front]



Multimodality in Multi-Objective Optimization?

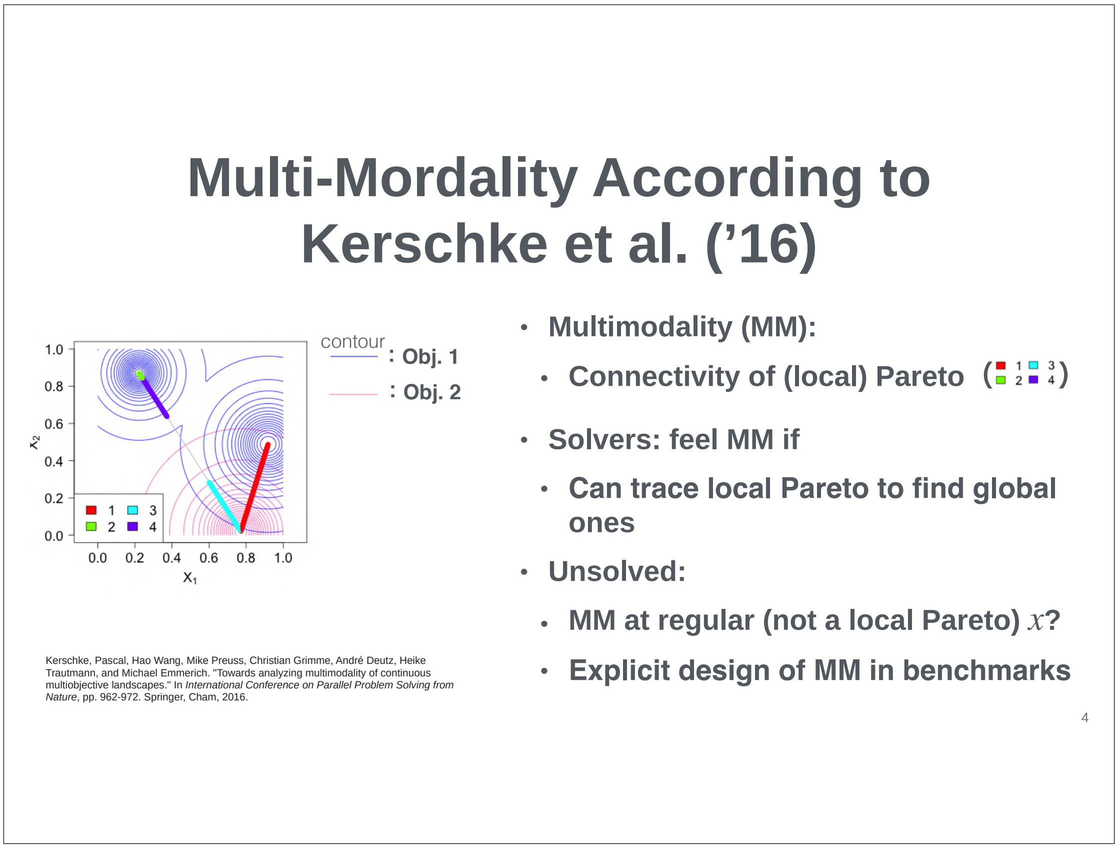
Single Objective Optimization

Multi-Objective Optimization

[wikimedia.org]

Multimodality:
- Depth / breadth of valleys
- Connectivity etc.

Multimodality:
- ?



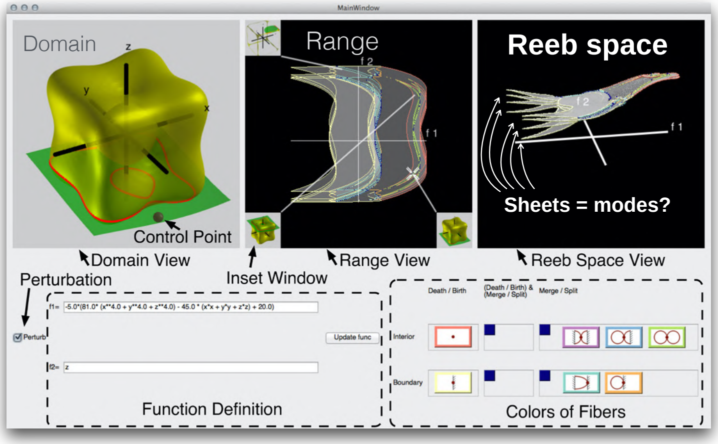
Multi-Mordality According to Kerschke et al. ('16)

Kerschke, Pascal, Hao Wang, Mike Preuss, Christian Grimme, André Deutz, Heike Trautmann, and Michael Emmerich. "Towards analyzing multimodality of continuous multiobjective landscapes." In *International Conference on Parallel Problem Solving from Nature*, pp. 962-972. Springer, Cham, 2016.

- Multimodality (MM):
  - Connectivity of (local) Pareto (🟥 1 🟦 3 🟩 2 🟪 4)
- Solvers: feel MM if
  - Can trace local Pareto to find global ones
- Unsolved:
  - MM at regular (not a local Pareto) x?
  - Explicit design of MM in benchmarks

Reeb Space and Multi-Modality

[Sakurai et al., 16]

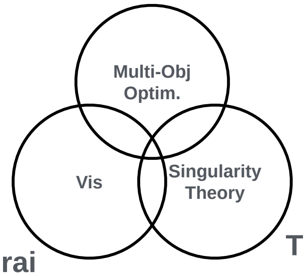


f1= -5.0*(81.0* (x**4.0 + y**4.0 + z**4.0) - 45.0 * (x*x + y*y + z*z) + 20.0)

f2= z

- UI for singular fibers (2016)
- Sheets = Multi-Modality?

Starting Collaboration



N Hamada

Multi-Obj Optim.

Vis

Singularity Theory

Sakurai

T Yamamoto
R Hagiwara
R Ota

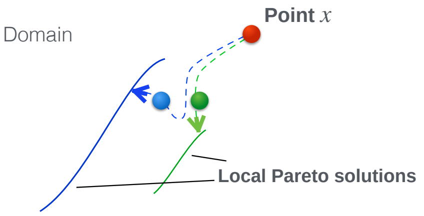
Model Multi-Modality

→ **Design Benchmark Problems (Work In Progress)**

- Jan 2021: rough ideas
- At present: formalization

Our Model of MM

- ***Admissible solution* at x :**
  - **Local Pareto component reachable by "optimizing" $f_i(x)$ (ie no sacrifice of any objective)**



Admissible at 🔴: {—, —}
Admissible at 🔵: {—}
Admissible at 🟢: {—}

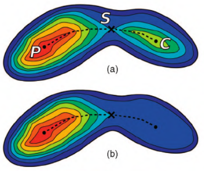
Hierarchy of feasibility

- ***Multimodality:* Characteristics of admissible solutions**

Hierarchy and Multi-Modality

Multi-modality in small scale

Multi-modality in large scale



[Weber et al., 07]

Also important in Vis

- **Can consider multiple scales**
- **Eg: evaluation of solvers:**
  - **Sensitivity to scales**
  - **Escape from small scale features?**

Designing Our Benchmark Problem

- **Polynomial maps：difficult to control singularity globally**
  → PL-maps
- $\mathbb{R}^3 \to \mathbb{R}^2$

Existing benchmarks: multi-modality unknown

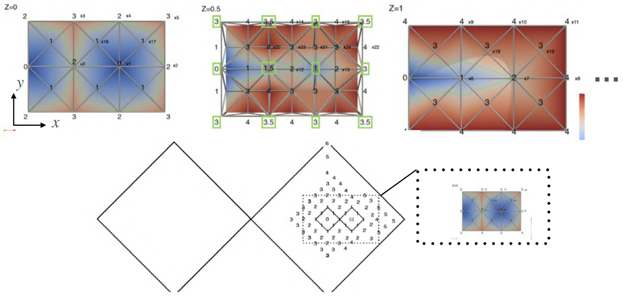
- 1990s: No standard problem
  - Researchers made their own problems
- Early 2000: ZDT [Zitzler+ 2000]
  - De facto standard problem set for 2 objectives
- Late 2000: DTLZ [Deb+ 2001], WFG [Huband+ 2006]
  - Extended ZDT for *n* objectives
- 2010s: Inverted-DTLZ [Lain+ 2013], Minus-WFG [Ishibuchi+ 2017]
  - Good methods for DTLZ/WFG don't work for negative costs

Primitives of Our Benchmark

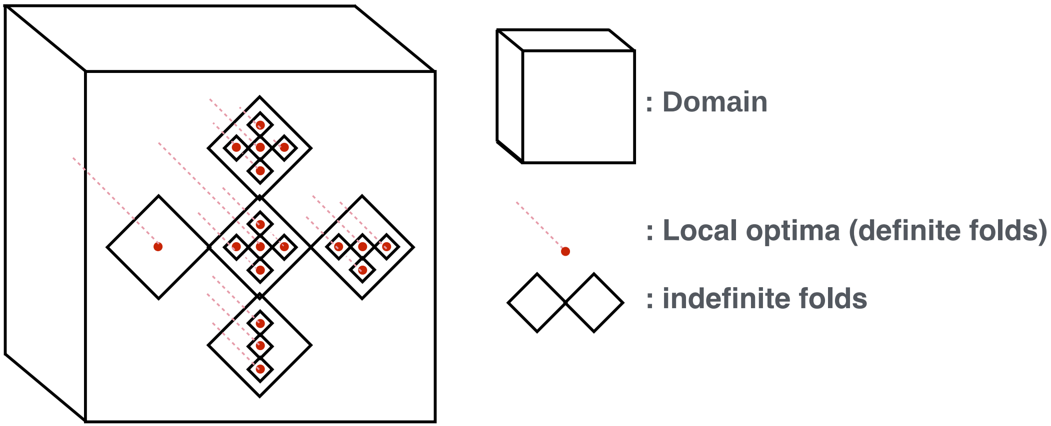
$$\begin{aligned} f_1 &= x^3 + zx + y^2 \\ f_2 &= z \end{aligned} \xrightarrow[\text{Non-linear transformation}]{\text{Discretization}} \text{Benchmark problem}$$

3 modes $z = 0$ $z = 0.5$ 1 mode $z = 1$ z

- **Primitives:**
  - **Local multi-obj optimization problem**



Forming Benchmark From Primitives



: Domain

: Local optima (definite folds)

: indefinite folds

Outlook

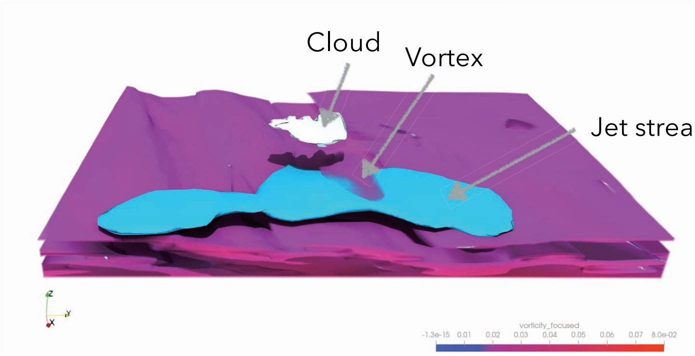
- Benchmarks for
  - $R^3 \to R^2$
  - $R^2 \to R^2$
- Latest manuscript in arxiv.org

The Centroids of Isosurface Components for All Isovalues

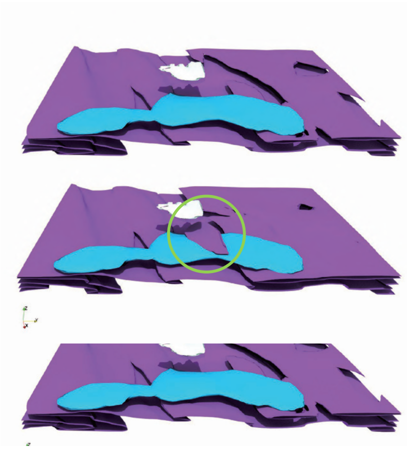
Akito Fujii
Daisuke Sakurai
Kenji Ono
(Kyushu University)

Analyzing multiple fields

Eg analysis with meteorologists:
Find relative positions of isosurface components from 3 scalar fields



Subjectivity in isosurfacing



Vorticity	Hypothesis (Vortex between cloud & jet)
0.019	**Stands not**
0.018	**Stands**
0.017	**Stands not**

Biases in visualization-based analysis

- Validity of hypotheses is biased by user inputs

- Multiple fields
- Multiple isovalues

- Analyze with which isovalues user specified hypotheses on relative positions stand (future research idea)

Preparation (presented today): compute centroid positions for all isovalues in scalar field

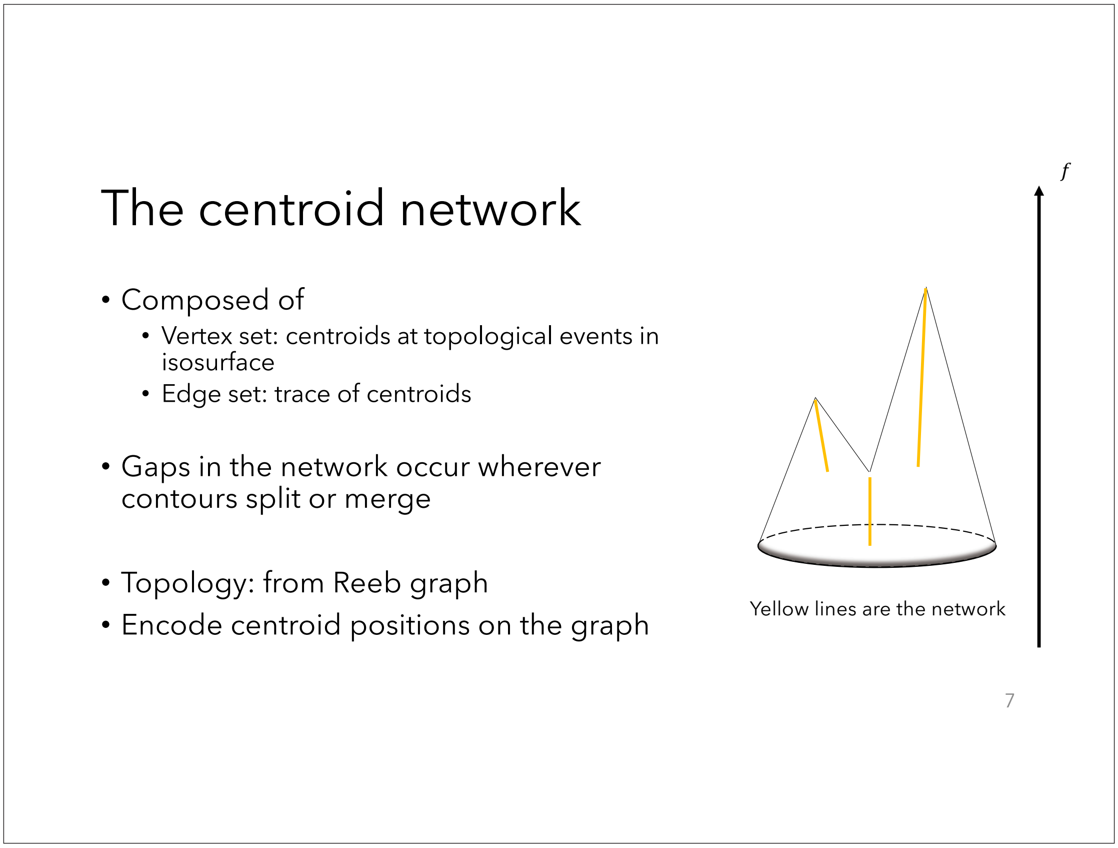
- Compute the centroids of isosurface components
- Use it in the future for analyzing relative positions in multiple fields

Computing geometrical information of isosurfaces

- Bajaj et al. 1997
  - Computed some values for each isosurface as a whole
    - length, 2D area, 3D volume etc.

- Carr et al. 2010
  - Computed geometrical values for isosurface components
  - For contour trees

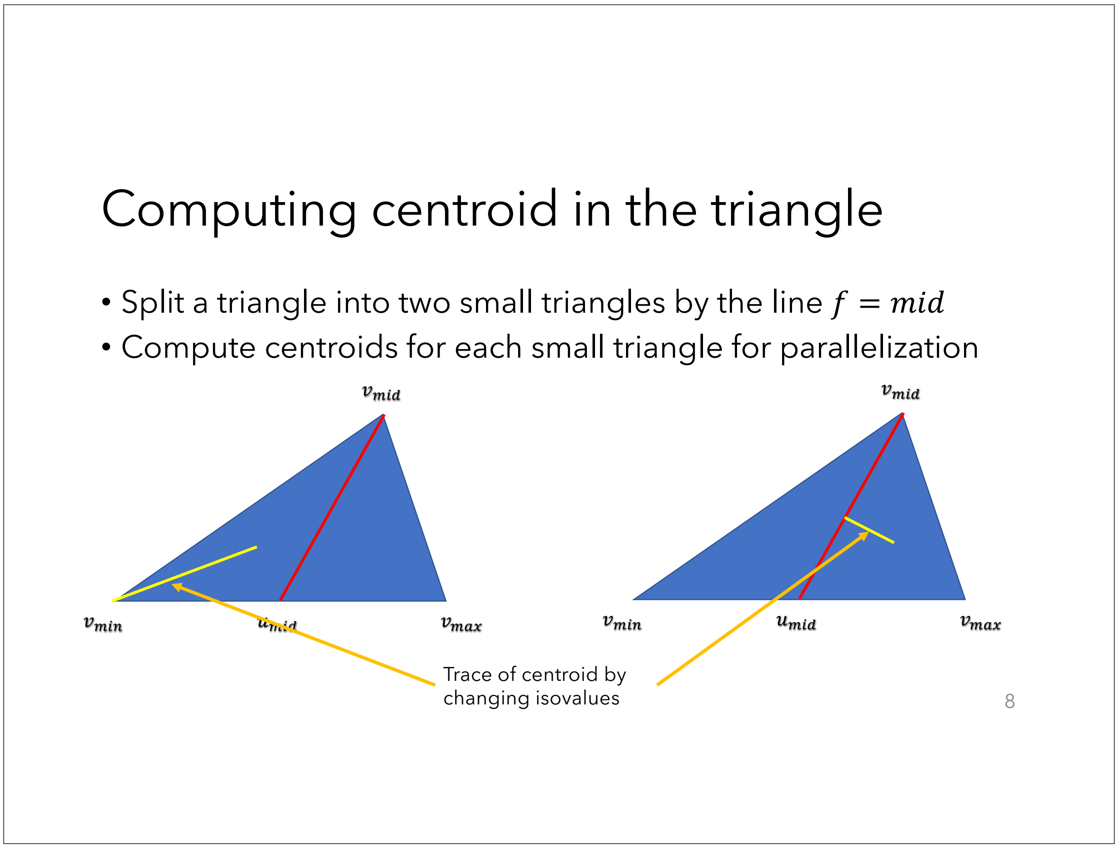
- Ours
  - Compute positions for manifold domains
  - Easy implementation
  - Insert few lines to existing Reeb graph implementation (of [Parsa 2012])



The centroid network

- Composed of
  - Vertex set: centroids at topological events in isosurface
  - Edge set: trace of centroids
- Gaps in the network occur wherever contours split or merge
- Topology: from Reeb graph
- Encode centroid positions on the graph

Yellow lines are the network



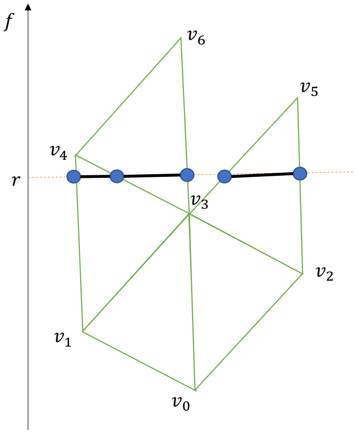
Computing centroid in the triangle

- Split a triangle into two small triangles by the line $f = mid$
- Compute centroids for each small triangle for parallelization

Trace of centroid by changing isovalues

[Parsa, 2012]

Preimage graph: encodes contour connectivity $(f(v_3) < r < f(v_4))$



- Nodes= $\{|v_1v_4|, |v_3v_4|, |v_3v_6|, |v_3v_5|, |v_2v_5|\}$
- Arcs = $\{|v_1v_3v_4|, |v_3v_4v_6|, |v_2v_3v_5|\}$

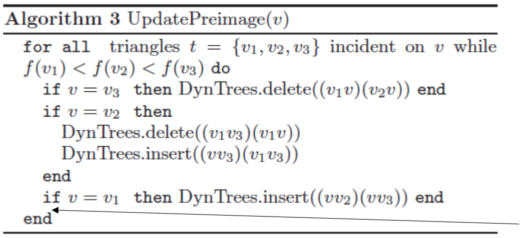
- Keep the transition of connected components of contours by changing r

Alteration to pseudo code

```
CreateNetwork(mesh)   vertices are sorted in the increasing order.
Preimage_graph is the graph which has vertices but no edges.
Network is the graph which has no vertices and edges.
Triangles = []                    # initialization
for vertex in mesh:
    G = 0, S = 0
    Lc = LowerComps(vertex, Preiamge_graph, Triangles)
    if Lc != []:     ComputeCentroid(vertex, Lc, Triangles)
    UpdatePreimage(vertex, Preimage_graph, Triangles)
    Uc = UpperComps(vertex, Preimage_graph, Triangles)
    if Uc != []:   ComputeCentroid(vertex, Uc, Triangles)
    UpdateNetwork(Lc, Uc, Network)
Return Network
```

Subroutines of creating the Reeb graph [Parsa, 2012]

UpdatePreimage(vertex, Preimage_graph, Triangles)



```
Algorithm 3 UpdatePreimage($v$)
  for all triangles $t = \{v_1, v_2, v_3\}$ incident on $v$ while $f(v_1) < f(v_2) < f(v_3)$ do
    if $v = v_3$ then DynTrees.delete($(v_1 v)(v_2 v)$) end
    if $v = v_2$ then
      DynTrees.delete($(v_1 v_3)(v_1 v)$)
      DynTrees.insert($(v v_3)(v_1 v_3)$)
    end
    if $v = v_1$ then DynTrees.insert($(v v_2)(v v_3)$) end
  end
```

Triangles.add(t)

[Parsa, 2012]

Compute and update preimage graph

Current situation

- I’m implementing now
- VTK + Python

Thinking multi-objective optimization problem topologically

Starting from hyperparameters

Likun Liu, Kenji Ono, Daisuke Sakurai

(Kyushu University)

Overview

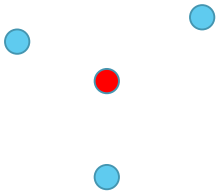
- Theory basis
  - Simplicial problem
  - Data abstraction
  - Space partitioning
- Proposed Method
- Current progress

Simplicial problem

- Definition
  - A multi-objective optimization problem is C^r simplicial if the Pareto set and the Pareto front are C^r diffeomorphic to a simplex and, under the C^r diffeomorphisms, each face of the simplex corresponds to the Pareto set and the Pareto front of a subproblem, where $0 \leq r \leq \infty$
- C^r is a topology space that met certain criteria
- Diffeomorphic: The mapping is bidirectional and is r-th order differentiable

---

Simplicial Optimization Problems[1]



🔵 : model parameters trained

🔴 : interpolate model without training

- For simplicial problems, training results can be approximated
- No re-training required
- The determination of the simplicial problem

[1] K. Kobayashi, N. Hamada, A. Sannai, A. Tanaka, K. Bannai, and M. Sugiyama, "Bézier Simplex Fitting: Describing Pareto Fronts of´ Simplicial Problems with Small Samples in Multi-Objective Optimization," *Proc. AAAI Conf. Artif. Intell.*, vol. 33, no. 01, pp. 2304-2313, 2019.

Determination – None Simplicial[2]

- Defination
  - Let $\mathrm{X}^*(f)$ be the pareto set of function $f = (f_1, \dots, f_m)$
  - Let $\Delta^{m-1} := \{w \in [0,1]^m \mid \sum_i w_i = 1\}$
  - 1) Exist Homeomorphic Mapping $\emptyset$: $\Delta^{m-1} \rightarrow \mathrm{X}^*(f)\ \mathrm{s.t.}\ \emptyset(\Delta_I) = \mathrm{X}^*(f_I)(\forall I)$
  - 2) Restrict Mapping $f|_{\mathrm{X}^*(f_I)} \rightarrow \mathbb{R}^m(\forall I)$ is embeded in C^0
- It's difficult to prove if $\emptyset$ exist, 1) can be interpreted as:
  - $\mathrm{X}^*(f_I)$ and $\Delta^{|I|-1}$ is homeomorphic
  - $int\mathrm{X}^*(f_I) \cap int\mathrm{X}^*(f_J) = \emptyset$
- If Homology group $H_*(A)$ and $H_*(B)$ are not isomorphism, tha space A and B are not homeomorphic
  - Prove that exist homology $q\ s.t. H_q\left(\mathrm{X}^*(f_J)\right) \ncong H_q\left(\Delta^{|I|-1}\right)$

[2] 濱田 直希, "多目的最適化の解集合のトポロジーの検定法," in *日本応用数理学会2020年年会*, 2020, pp. 267–268.

Determination – Simplicial[2]

- Let M be a compact C^∞ manifolds
  - If M and ∂M are both simply connected space
  - $H_q(M) \cong H_q(\Delta^n) \cong \begin{cases} \mathbb{Z}\ (q = 0) \\ 0\ (q \neq 0) \end{cases}$
  - M and Δ^n are Homeomorphic

[2] 濱田 直希, "多目的最適化の解集合のトポロジーの検定法," in *日本応用数理学会2020年年会*, 2020, pp. 267–268.

Space partitioning

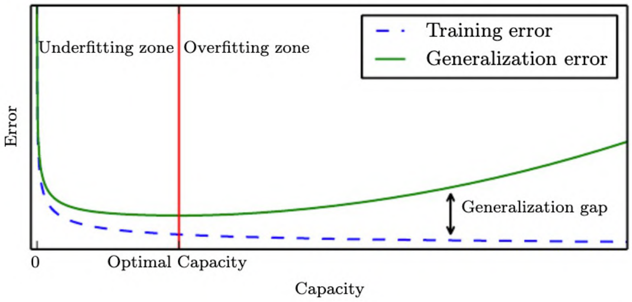
Data structure	Applicable situations	Grade of N	Time complexity	Dynamically updatable	Storage Required
Quad Tree	Scene management without height, rendering	Nmuber of objects	$O(nlogn)$	Yes	Large (Depend on area size and number of objects)
Oct Tree	Scene management, rendering	Nmuber of objects	$O(nlogn)$	Yes	Large (Depend on area size and number of objects)
BVH Tree	Physics, rendering (almost anything)	Number of objects	$O(nlogn)$	Yes	Medium (Depend on the number of objects)
BSP Tree	Editor, complex scene	Number of planes	$O(nlog^2n)$	No	Large (Depend on the number of planes)
Accerated BSP tree with BVH	Editor, complex scene	Number of planes	$O(nlogn)$	Partially Yes	Large (Depend on area size and number of objects)

Validation – Staring from ML

- Generally, in algorithms, we have parameters that controlled the behavior of the algorithm
  - SVM -> Kernel
  - K-NN -> k
  - Loose Oct/Quad Tree -> expansion factor k
- The raise of the ML has made the selection of the hyperparameters important
  - Running time and storage costs
  - Model quality
  - Inference accuracy
  - *Model Architecture

Manual hyperparameter tuning

- To set hyperparameters manually…
  - Type of hyperparameters
  - Training error
  - Generalization error
- When plotting generalization error regarding a hyperparameter



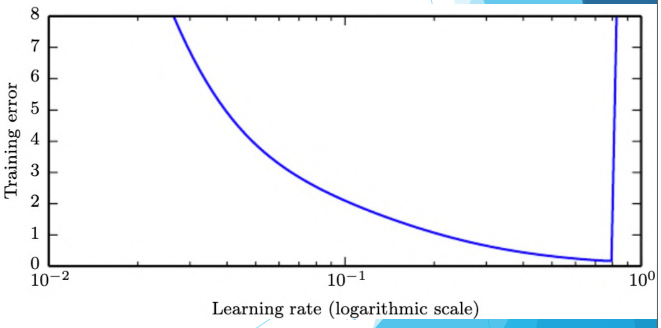
Model capacity:
The ability to represent complex functions of the model

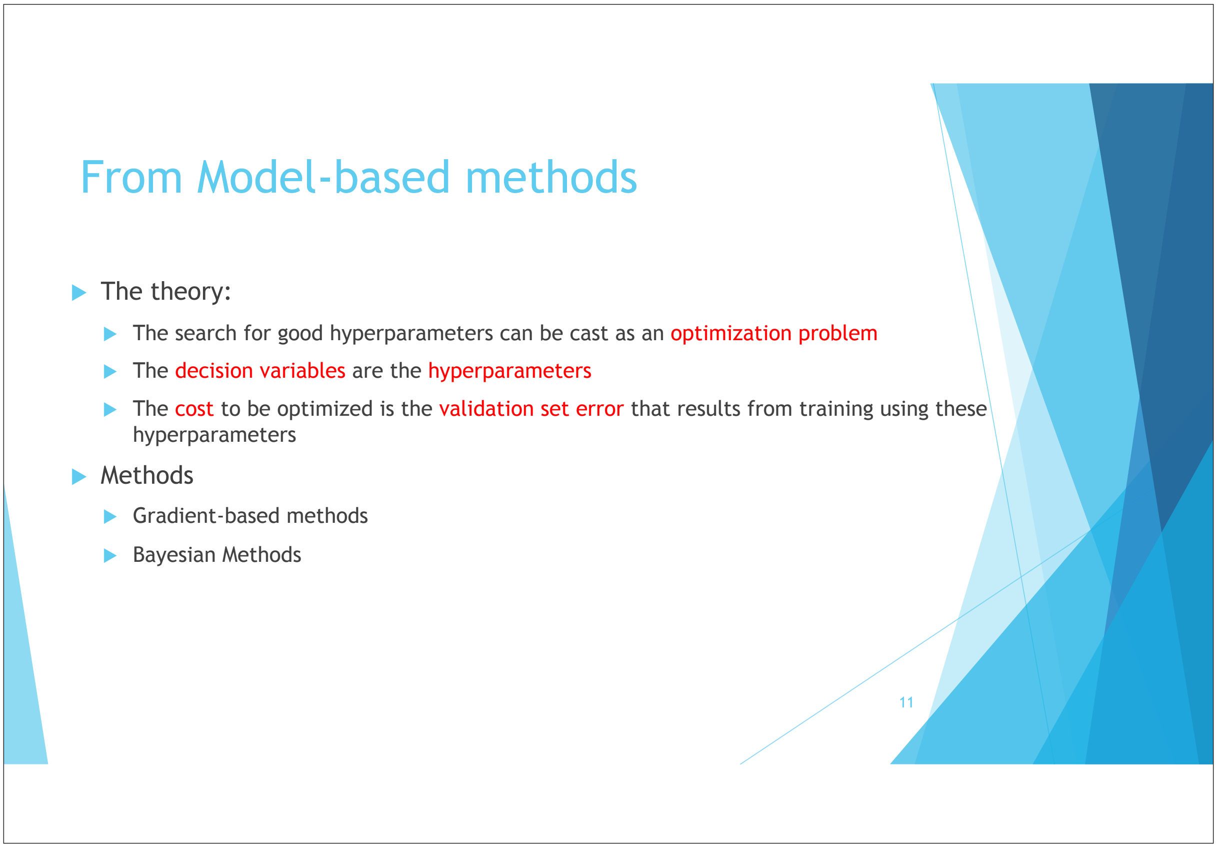
Effective capacity:
The capacity that actually used to represent functions

Examples

- Number of hidden unit :
  - Increase the hidden unit can increase the representational capacity of the model
  - Increase the time and memory cost
- Learning rate:
  - Controls the effective capacity of the model
  - Have an optimal value for optimization problem
  - * If you only have time for one hyperparameter,

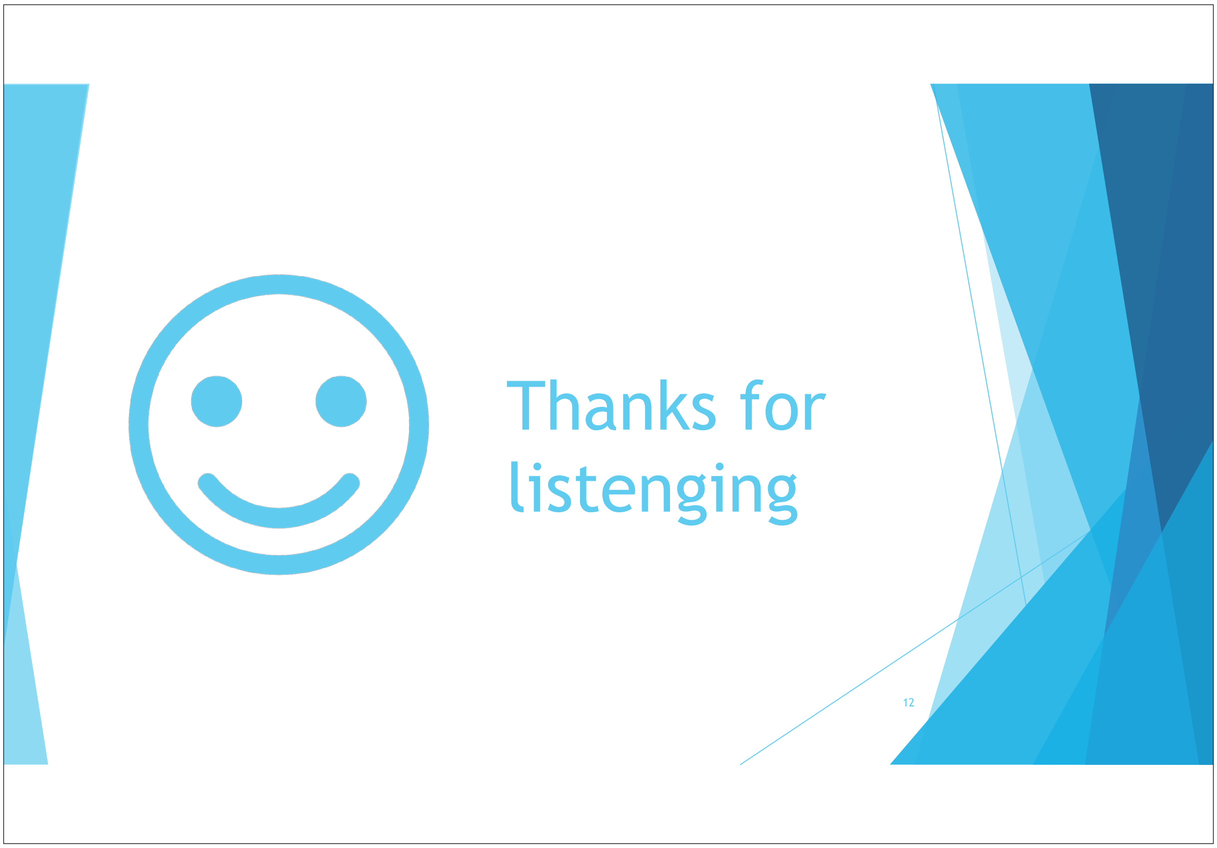
    Go for Learning Rate



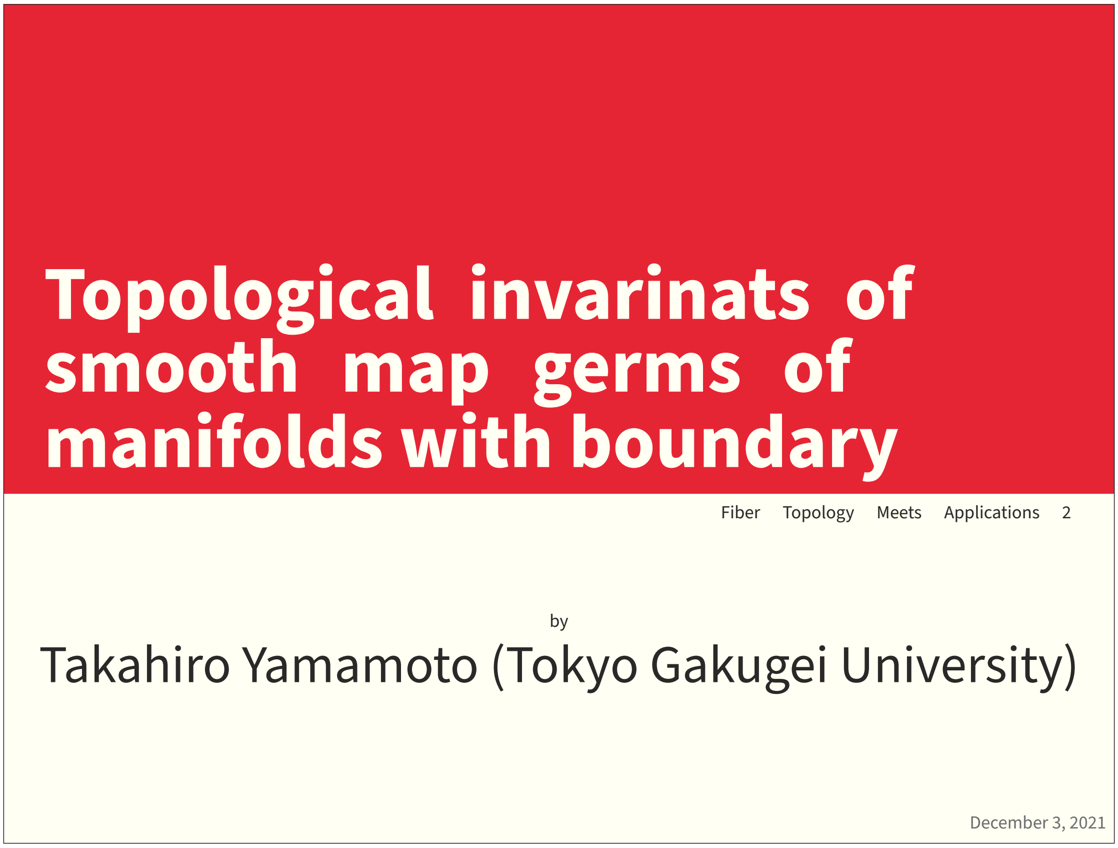


From Model-based methods

- The theory:
  - The search for good hyperparameters can be cast as an optimization problem
  - The decision variables are the hyperparameters
  - The cost to be optimized is the validation set error that results from training using these hyperparameters
- Methods
  - Gradient-based methods
  - Bayesian Methods



Thanks for listenging



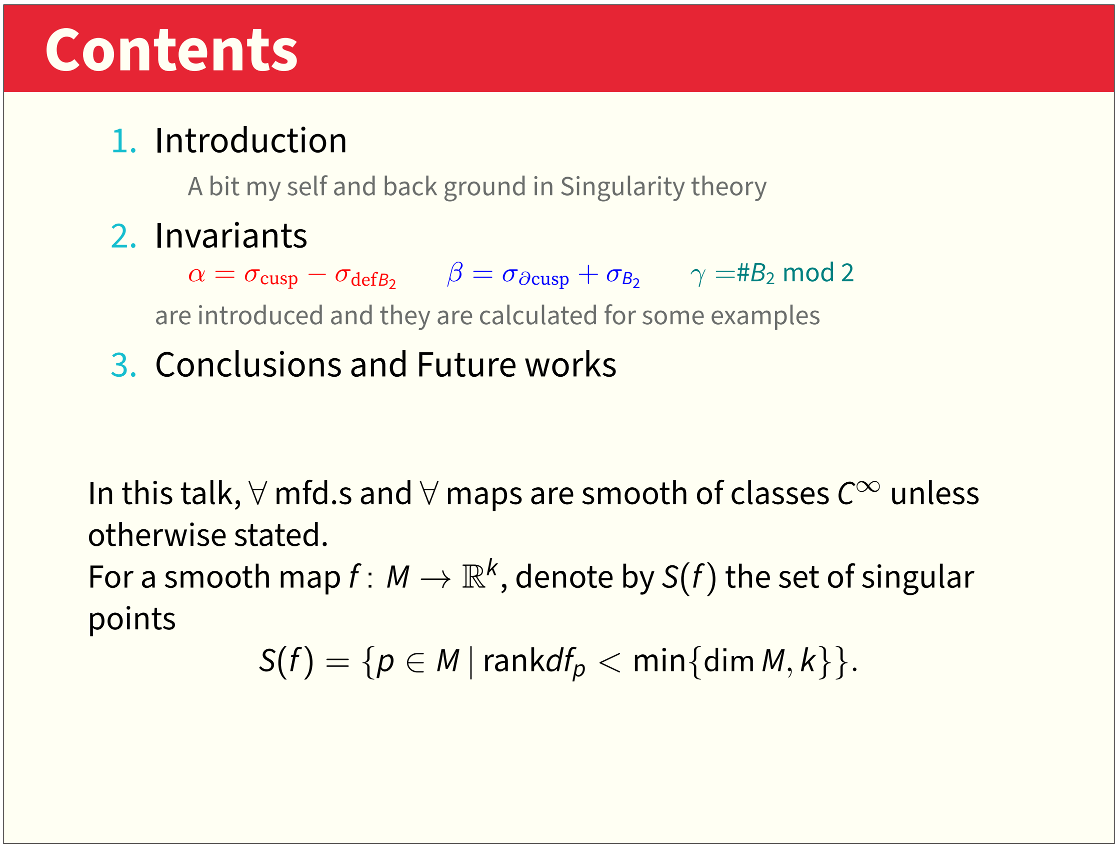
Topological invarinats of smooth map germs of manifolds with boundary

Fiber Topology Meets Applications 2

by

Takahiro Yamamoto (Tokyo Gakugei University)

December 3, 2021



Contents

1. Introduction
   A bit my self and back ground in Singularity theory
2. Invariants
   $\alpha = \sigma_{\mathrm{cusp}} - \sigma_{\mathrm{def}B_2}$ $\beta = \sigma_{\partial\mathrm{cusp}} + \sigma_{B_2}$ $\gamma = \#B_2 \bmod 2$
   are introduced and they are calculated for some examples
3. Conclusions and Future works

In this talk, $\forall$ mfd.s and $\forall$ maps are smooth of classes C^∞ unless otherwise stated.
For a smooth map $f\colon M \to \mathbb{R}^k$, denote by $S(f)$ the set of singular points

$$S(f) = \{p \in M \mid \mathrm{rank} df_p < \min\{\dim M, k\}\}.$$

Introduction

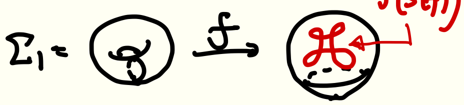
Takahiro Yamamoto

INTEREST: To study TOPOLOGY and GEOMETRY of spaces M, in particular manifolds, by using singularity of smooth maps $f\colon M \to \mathbb{R}^k$, $(\dim M \geq k)$.

KEY WORDS: Singularity of smooth maps, Stable Maps, Generic Maps, Singular Fibers

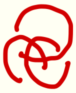
FAVORITE FORMULAS: [Kamenosono -**Y**('08)] For a stable map $f\colon \Sigma_g \to S^2$, we have

$$g = N + \frac{c}{2} + (1 + l) - m(f).$$



[Saeki -**Y**('06)] For a stable map $f\colon M \to \mathbb{R}^3$ of a closed oriented 4-mfd, we have

$$\sigma(M) = ||\underbrace{\mathrm{III}^8(f)}||.$$

alg. # of singular fiber of type 

Joint Works with Computer Sci.

(By O. Saeki, S. Takahashi, D. Sakurai, H. Wu, K. Kikuchi, H. Carr, D. Duke, **Y**)
(1) The Impact of Applications on Mathematics, Proceedings of Forum of Mathematics for Industry 2013, 51–65.
(2) IEEE TRANSACTIONS ON VISUALIZATION AND COMPUTER GRAPHICS, VOL. 22, NO. 1, 945–954, 2016.
! Contributed by classifying singular fibers of stable maps $M^3 \to \mathbb{R}^2$ of cpt 3-mfds with boundary.

(D. Sakurai, **Y**)
(3) Topological Methods in Data Analysis and Visualization VI, 181–196, 2021.
! Contributed by constructing the theory and calculating example by using "jcnet".

(By R. Hagiwara, R. Ota, N. Hamada, D. Sakurai, **Y**)
(4) Explicitly Multimodal Benchmarks for Multi-Objective Optimizations

Back ground in Singularity thoery

Classifying mappings or map germs is an important and fundamental problem in Singularity theory.
! In order to classify mappings or map germs, we need **INVARIANTS**. It implies that studying invariants are important.

For a given map germs, invariants which are calculated by perturbing the map germs stably are obtained in SOME cases:

Milnor ('68)	$g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$,
Fukuda-Ishikawa ('87)	$g\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$,
$\vdots$	
Saeki ('07)	$g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, and
Y ('18, 19)	$g\colon (H^n, 0) \to (\mathbb{R}^2, 0)$ $\leftarrow$ **Today!!**

! It is **NOT** easy obtaining stable map germs by deforming $\{g_t\}$ a given map germ g at the formula level, $g_t = g + tf$ for example.

Today

For map germs of 3-mfd.s with ∂ into $\mathbb{R}^2$, there are three invariants

$$\alpha = \sigma_{\text{cusp}} - \sigma_{\text{def}B_2} \qquad \beta = \sigma_{\partial\text{cusp}} + \sigma_{B_2} \qquad \gamma = \#B_2 \text{ mod } 2$$

calculated by perturbing the map germs stably.

In this talk, we show that for a given map germ, we can perturb the map germ stably by using **the fiber topology and its visualization software**. Then, the invariants are actually **calculated by using the obtained stable map germs**.

! By calculating α, β, γ in some cases, we estimate how strong they are.

Invariants

Map germs $(H^n, 0) \to (\mathbb{R}^2, 0)$

Denote by H^n the upper half subset of $\mathbb{R}^n$, namely

$$H^n := \{(x_1, \dots, x_{n-1}, y) \in \mathbb{R}^n \mid y \geq 0\},$$

where $\{y = 0\}$ corresponds to **the bdry of our mfd**, whose **interior** is taken to be the part of $\mathbb{R}^n$ **with** $y > 0$.

Equivalence relation among map germs $(H^n, 0) \to (\mathbb{R}^k, 0)$

$g, g' \colon (H^n, 0) \to (\mathbb{R}^k, 0)$: map germs.
$g \sim g'$: **topologically $\mathcal{B}_+$-equivalent**
$\overset{\text{def}}{\Leftrightarrow} \exists \Phi \colon (H^n, 0) \to (H^n, 0)$: homeo. preserving $\{y = 0\}$, $\{y > 0\}$,
$\exists \psi \colon (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$: homeo. preserving orientation
s.t. $g \circ \Phi = \psi \circ g'$.

Stable maps

$f \colon M \to \mathbb{R}^k$: smooth
f is **stable** $\overset{\text{def}}{\Leftrightarrow} \exists N(f) \subset C^\infty(M, \mathbb{R}^k)$: a nbd of f s.t. $\forall f' \in N(f)$, f and f' are equivalent, where f and f' are said equivalent if $\exists \Phi \colon M \to M$: diffeo. preserving ∂, $\exists \psi \colon \mathbb{R}^k \to \mathbb{R}^k$: diffeo. s.t. they make the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\Phi"] \arrow[d, "f"'] & M \arrow[d, "f'"] \\
\mathbb{R}^k \arrow[r, "\psi"] & \mathbb{R}^k.
\end{tikzcd}$$

A map germ $g \colon (H^n, 0) \to \mathbb{R}^k$ is **stable** $\overset{\text{def}}{\Leftrightarrow}$ a representative $U \to \mathbb{R}^k$ of g is stable.

A stable map $f \colon M^3 \to \mathbb{R}^2$ is characterized by the following two (local and global) conditions.

Stable maps $M^3 \to \mathbb{R}^2$ (Local cond.)

A C^∞ map $f\colon M^3 \to \mathbb{R}^2$ is **stable** $\overset{\text{iff}}{\Leftrightarrow}$ f satisfies the conditions below:
(1) For $\forall p \in \mathrm{Int}M$, (f, p) is equiv. to

$$(x_1, x_2, y) \mapsto \begin{cases} (x_1, x_2) & p \text{ is a reg. pt, or} \\ (x_1, x_2^2 \pm y^2) & p\text{: a def/indef fold pt,} \\ (x_1, x_2^3 + x_1x_2 + y^2) & p\text{: a cusp pt.} \end{cases}$$

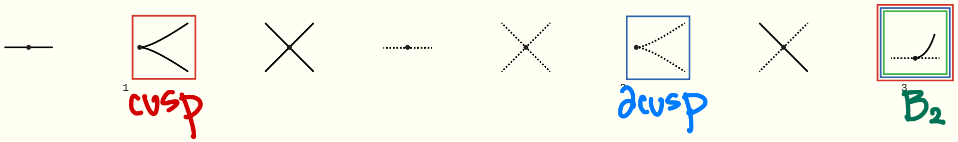
For $\forall p \in \partial$, (f, p) is equiv. to

$$(x_1, x_2, y) \mapsto \begin{cases} (x_1, x_2) & p \text{ is a reg. pt of } f|_{N(\partial)}\text{, or} \\ (x_1, x_2^2 \pm y) & p\text{: a def/indef } \partial\text{fold pt,} \\ (x_1, x_2^3 + x_1x_2 + y) & p\text{: a } \partial\text{cusp pt,} \\ (x_1, x_2^2 \pm y^2 + x_1 y) & p\text{: a def/indef } B_2 \text{ pt,} \end{cases}$$

where $\mathrm{Int}M \Leftrightarrow \{y > 0\}$ and $\partial \Leftrightarrow \{y = 0\}$.

Stable maps $M^3 \to \mathbb{R}^2$ (Global cond.)

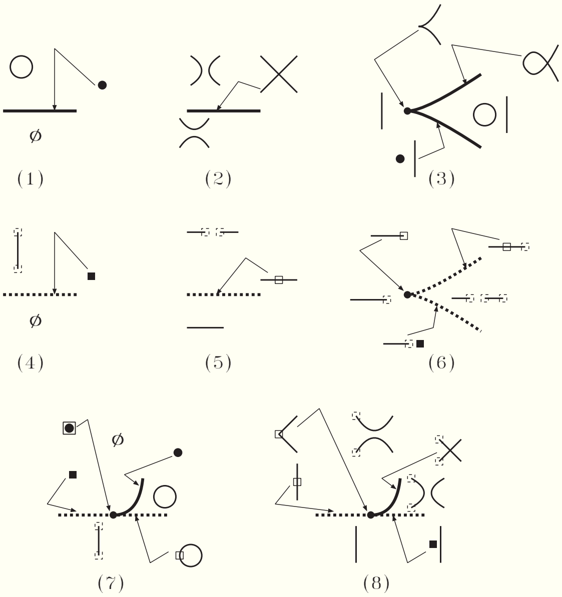
A C^∞ map $f\colon M^3 \to \mathbb{R}^2$ is **stable** $\overset{\text{iff}}{\Leftrightarrow}$ f satisfies the conditions below:
(1) . . .
(2) For $\forall q \in f(S(f) \cup S(f|_\partial))$, $(f(S(f) \cup S(f|_\partial)), q)$ is equivalent to one of the figures below:



where solid lines $\Leftrightarrow$ $f(S(f))$ and dotted lines $\Leftrightarrow$ $f(S(f|_\partial))$.

! B_2 is fold (or fold) if we ignore ∂ (resp. restrict f to ∂).
! For a stable map $f\colon M^3 \to Q^2$, the pre-image $f^{-1}(q)$ of a reg value consists of finite number of circles and arcs.

Local degeneration of level sets



(1) (2) (3)

(4) (5) (6)

(7) (8)

Fibers of smooth maps

For a C^∞ map $f\colon M \to Q$ and $q \in Q$, the map germ

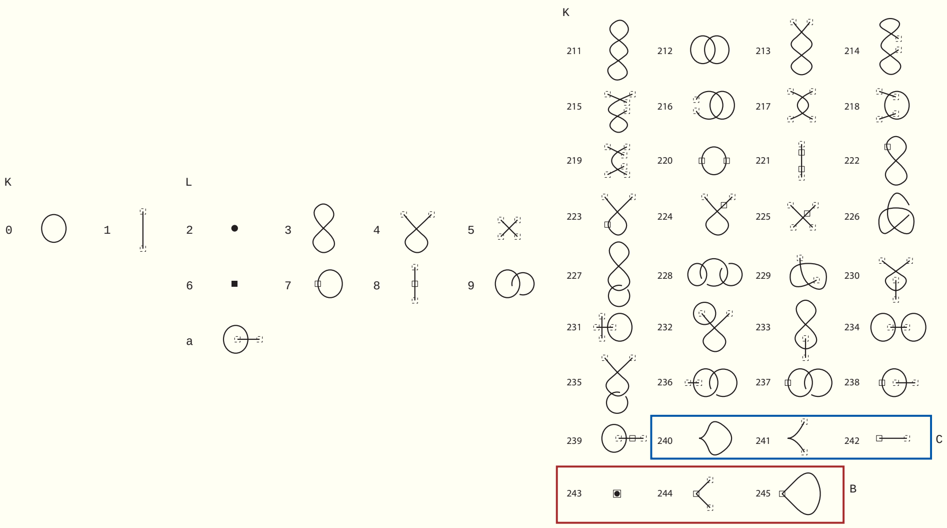
$$f\colon (M, f^{-1}(q)) \to (Q, q)$$

is called the **fiber** over $q \in Q$.

Fibers $f_i\colon (M_i, f_i^{-1}(q_i)) \to (Q_i, q_i)$ $(q_i \in Q_i)$ are **C^0-equiv** if
$\exists U_i \subset Q_i$: a nbd of $q_i \in Q$, $(i = 0, 1)$,
$\exists\, \Phi\colon (f_0^{-1}(U_0), f_0^{-1}(r_0)) \to (f_1^{-1}(U_1), f_1^{-1}(q_1))$: homeo preserving ∂
$\exists\, \varphi\colon U_0 \to U_1$: homeo with $\varphi(q_0) = q_1$
s.t. they make the following diagram commutative:

$$\begin{tikzcd}
(f_0^{-1}(U_0), f_0^{-1}(q_0)) \arrow[r, "\Phi"] \arrow[d, "f_0"'] & (f_1^{-1}(U_1), f^{-1}(q_1)) \arrow[d, "f_1"] \\
(U_0, q_0) \arrow[r, "\varphi"] & (U_1, q_1).
\end{tikzcd}$$

Fibers of $M^3 \to Q^2$ of 3-mfd.s with ∂



Invariants

THM 1 [Y'18] ($n = 3$)

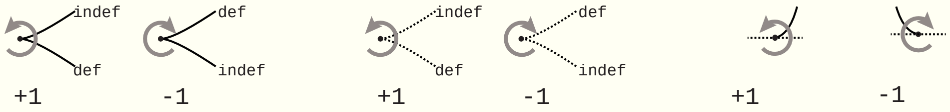
$g\colon (H^3, 0) \to (\mathbb{R}^2, 0)$: a "cone-like"map germ Then,
for a stable map $f\colon U \to \mathbb{R}^2$ obtained by perturbing a representative $U \to \mathbb{R}^2$ of g stably,
(1) $\alpha(f) = \sigma_{\mathrm{cusp}}(f) - \sigma_{\mathrm{def}B_2}(f)$ (2) $\beta(f) = \sigma_{\partial\mathrm{cusp}}(f) + \sigma_{B_2}(f)$
(3) $\gamma(f) = \#B_2(f)$ mod 2
are topological $\mathcal{B}_+$-invariants, where $\sigma_* =$ "the alg. # of singularities of type $*$".

! For a generic map germ $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ and a generic hyperplane $\mathcal{H} \subset \mathbb{R}^n$, a map germ g restrict to the upper half space of $\mathcal{H}$ is "conelike".

Signs of cusps and ∂cusps, B_2

$f\colon M \to \mathbb{R}^2$: a stable map of a cpt 3-mfd w/ ∂
p: a cusp or a ∂cusp, a B_2 point of f

Orient $\mathbb{R}^2$ in the counter-clockwise manner.
If the image around $f(p)$ are in the left hand side of



then sign$(p) = +1$; otherwise, sign$(p) = -1$.

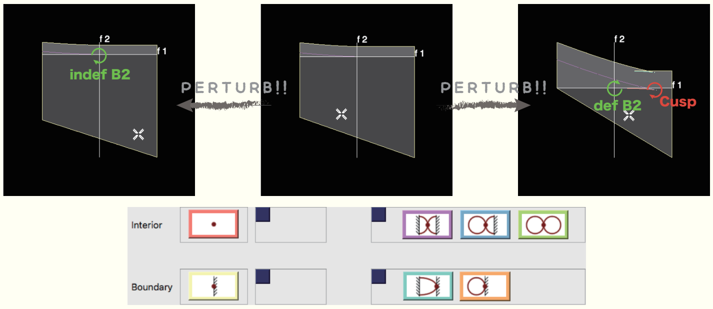
σ_{cusp} (or $\sigma_{\partial\text{cusp}}$, $\sigma_{\text{def}B_2}$ σ_{B_2}) is the total of signs of cusps (resp. ∂cusps, definite B_2, B_2) of f.

Example 1: B_{3+} (well-known)

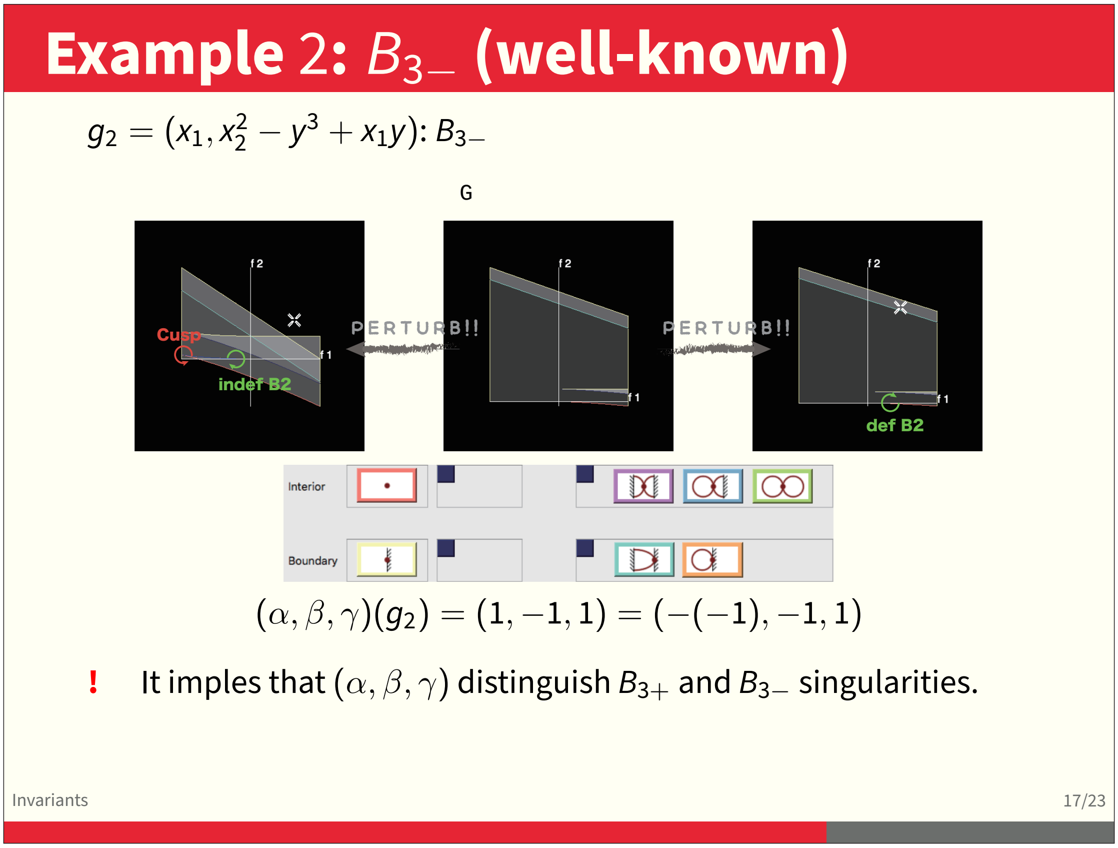
! In the following, g_i is perturbed by a software "jcnet"which draws the discriminant (and the Reeb space) of g_i. Furthermore "jcnet"can perturb a given map germ by mouse interaction.

$$g_1 = (x_1, x_2^2 + y^3 + x_1 y)\colon B_{3+}$$

G



$$(\alpha, \beta, \gamma)(g_1) = (0, -1, 1) = (-1 - (-1), -1, 1)$$

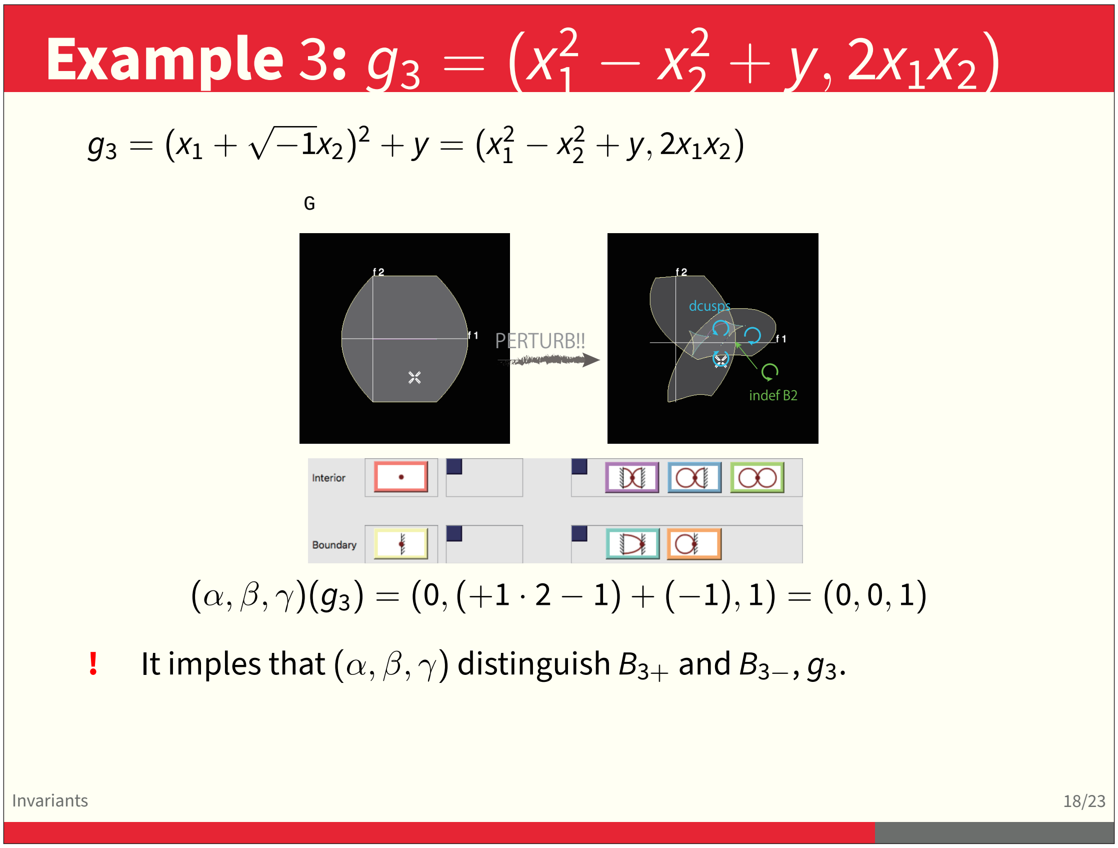


Example 2: B_{3-} (well-known)

$g_2 = (x_1, x_2^2 - y^3 + x_1 y)$: B_{3-}

$$(\alpha, \beta, \gamma)(g_2) = (1, -1, 1) = (-(-1), -1, 1)$$

! It imples that (α, β, γ) distinguish B_{3+} and B_{3-} singularities.



Example 3: $g_3 = (x_1^2 - x_2^2 + y, 2x_1x_2)$

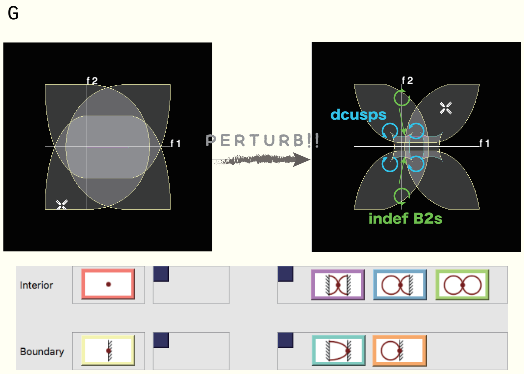
$g_3 = (x_1 + \sqrt{-1}x_2)^2 + y = (x_1^2 - x_2^2 + y, 2x_1x_2)$

$$(\alpha, \beta, \gamma)(g_3) = (0, (+1 \cdot 2 - 1) + (-1), 1) = (0, 0, 1)$$

! It imples that (α, β, γ) distinguish B_{3+} and B_{3-}, g_3.

Example 4: $g_4 = (x_1^3 - 3x_1x_2^2 + y, 3x_1^2x_2 - x_2^3)$

$$g_4 = (x_1 + \sqrt{-1}x_2)^3 + y = (x_1^3 - 3x_1x_2^2 + y, 3x_1^2x_2 - x_2^3)$$

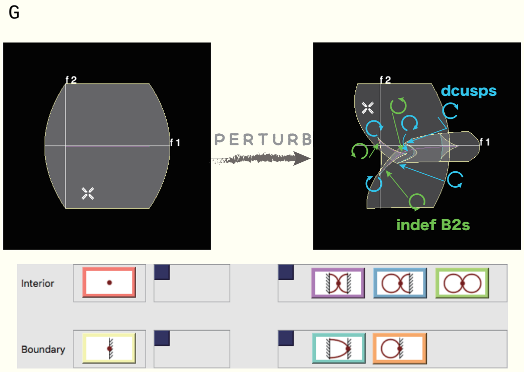


$$(\alpha, \beta, \gamma)(g_4) = (0, (+1 \cdot 2 - 1 \cdot 2) + (+1 - 1), 0) = (0, 0, 0)$$

! It imples that (α, β, γ) distinguish B_{3+} and B_{3-}, g_3, g_4.

Example 5: $g_5 = (x_1^2 - x_2^2 + y^2, 2x_1x_2)$

$$g_5 = (x_1 + \sqrt{-1}x_2)^2 + y^2 = (x_1^2 - x_2^2 + y^2, 2x_1x_2) \colon (X^3, 0) \to (\mathbb{R}^2, 0)$$



$$(\alpha, \beta, \gamma)(g_5) = (0, (+1 \cdot 1 - 1 \cdot 4) + (+1 \cdot 3 - 1 \cdot 0), 1) = (0, 0, 1)$$

! $(\alpha, \beta, \gamma)(g_3) = (0, 0, 1)$. Actually, g_3 and g_5 are topologically $\mathcal{B}_+$ equivalent.

Conclusions and Future Works

Conclusion

In this talk, we show that invariants α, β, γ are actually calculated via stable map germs obtained by using the fiber topology and its visualization software.

Q Can the calculation process be automated by computer sci.? Nemely,

$$\text{a map germ } g \xrightarrow[\textit{calculate}]{\textit{perturb stably}} (\alpha, \beta, \gamma)(g)$$

In particular, if perturbing stably process are automated, then, **HIDDEN** invariants might be discovered by studying a lot of examples!!

Future works

On the other hand, we have also obtained the invariants below:

THM 2 [Y'19] ($n \geq 2$)

$g\colon (H^n, 0) \to (\mathbb{R}^2, 0)$: a "cone-like"map germ Then, for a stable map $f\colon U \to \mathbb{R}^2$ obtained by perturbing a representative $U \to \mathbb{R}^2$ of g stably,

$$\gamma(f) = \#B_2(f) \bmod 2$$

is a topological $\mathcal{B}_+$-invariant.

! The invariant γ of a map germ g is calulated if we have a software like "jcnet"!!

Reeb Diagram and Visualization of Monodromy

Osamu Saeki
(Kyushu University)

December 3, 2021
Fiber Topology Meets Applications 2
IMI, Kyushu University

§1. Fiber

Setting

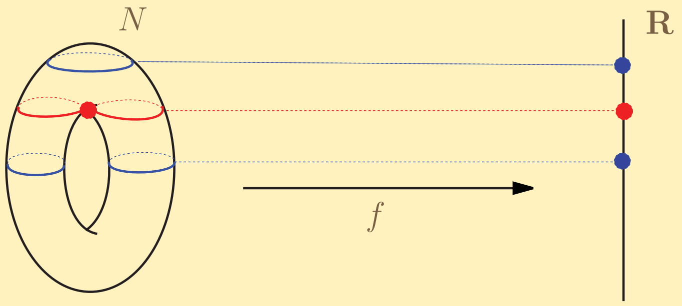
N^n, M^m : C^∞ **manifolds**, $f : N^n \to M^m$ C^∞ **map**
Usually, N^n is a bounded domain in $\mathbf{R}^n$.
Typically $M^m = \mathbf{R}^m$, and in this case

$$f = (f_1, f_2, \ldots, f_m) \qquad \textbf{multi-variate data}$$

We assume f is **generic** (C^∞ stable, C^0 stable, finite codimension, etc.)

We are interested in the topology of **fibers** $f^{-1}(y)$, $y \in M^m$.
Generically, $\dim f^{-1}(y) = n - m$. We usually assume $n \geq m$.

Example of fibers



We can grasp global feature of data by chasing fibers (or level sets).

We have some **singular fibers** (or **critical level sets**) where **topological transitions of fibers** occur.

Singular points and Jacobi set

$f : N^n \to M^m$ $(n \geq m)$ C^∞ map

Definition 1.1 For a point $p \in N^n$, consider the **differential**

$$df_p : T_pN^n \to T_{f(p)}M^m$$

(linear map associated with the **Jacobian matrix** of f).
Singular point is a point $p \in N^n$ with $\mathrm{rank}\ df_p < m$.
The set of singular points

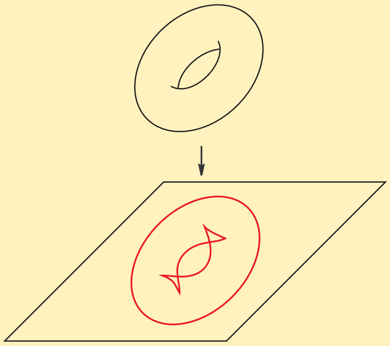
$$J(f) = \{p \in N^n \mid \mathrm{rank}\ df_p < m\}$$

is called the **Jacobi set** of f.

Generically, $\dim J(f) = m - 1$.

Jacobi set image

Jacobi set image $f(J(f))$ divides the range M^m into some regions.



Topology of fibers changes along the Jacobi set image.
Singular fiber is a fiber $f^{-1}(y)$ with $y \in f(J(f))$.
It is important to know topological changes of fibers near a singular fiber.

A formulation

Mathematically, a **fiber** is, in fact, NOT just a subset in the domain N^n, but a MAP around a pre-image.

$f_i : N_i^n \to M_i^m$ $\quad C^\infty$ maps, $i = 0, 1$
For points $y_i \in M_i^m$, $i = 0, 1$, fibers over y_0 and y_1 are **equivalent** if
∃commutative diagram

$$\begin{tikzcd}
(f_0^{-1}(U_0), f_0^{-1}(y_0)) \arrow[r, "\cong"] \arrow[d, "f_0"'] & (f_1^{-1}(U_1), f_1^{-1}(y_1)) \arrow[d, "f_1"] \\
(U_0, y_0) \arrow[r, "\cong"] & (U_1, y_1)
\end{tikzcd}$$

for some neighborhoods $y_i \in U_i \subset M_i^m$, $i = 0, 1$.

In particular, equivalent fibers have the **same topological transitions** of nearby fibers.

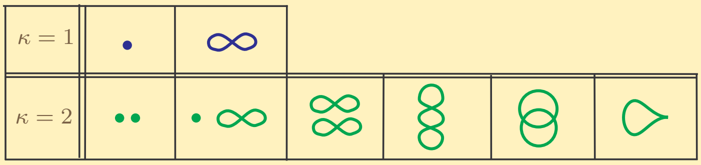
Classification of singular fibers

For certain dimensions, we can **classify singular fibers** of generic maps.

Example 1.2 *Classification results for* (n, m) *with* $n - m = 1$.
For simplicity, we assume the domain N^n *is* **orientable**.
We will ignore regular fiber components.

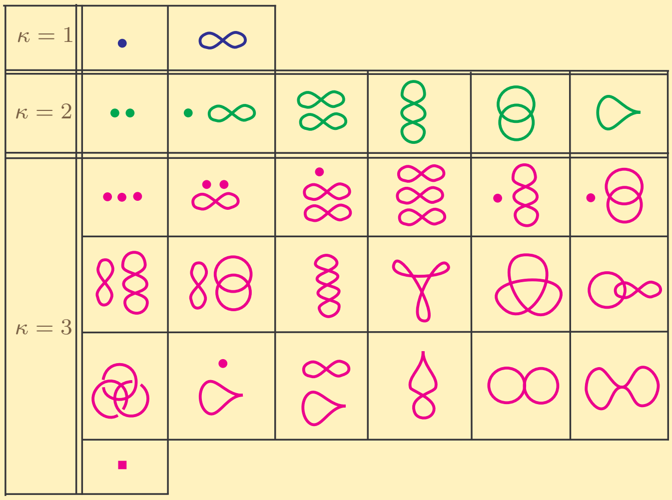
1. $(n, m) = (2, 1)$ [Folklore] $\quad \kappa = 1$ *(codimension)*

2. $(n, m) = (3, 2)$ [Kushner–Levine–Porto, 1984]



Singular fibers for $(n, m) = (4, 3)$

3. $(n, m) = (4, 3)$ [S.]



4. $(n, m) = (5, 4)$ [Yamamoto–S.]

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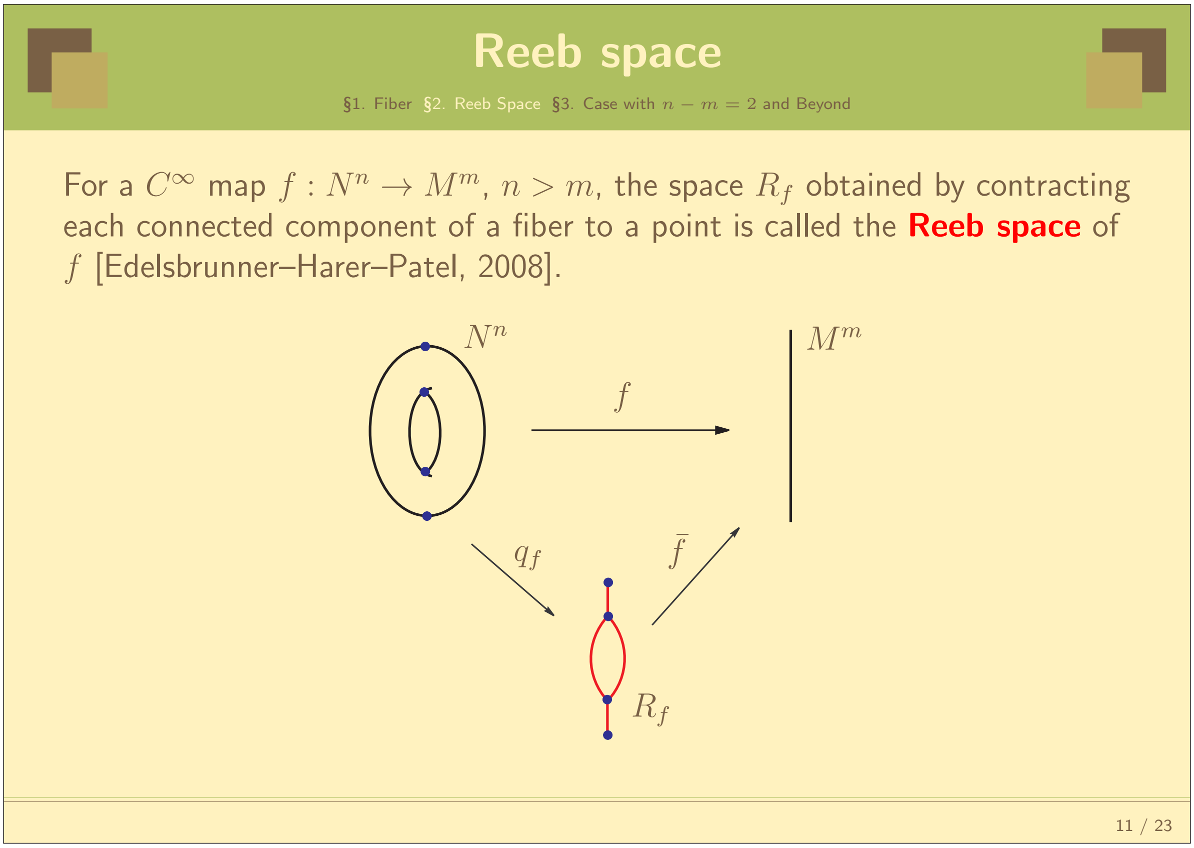
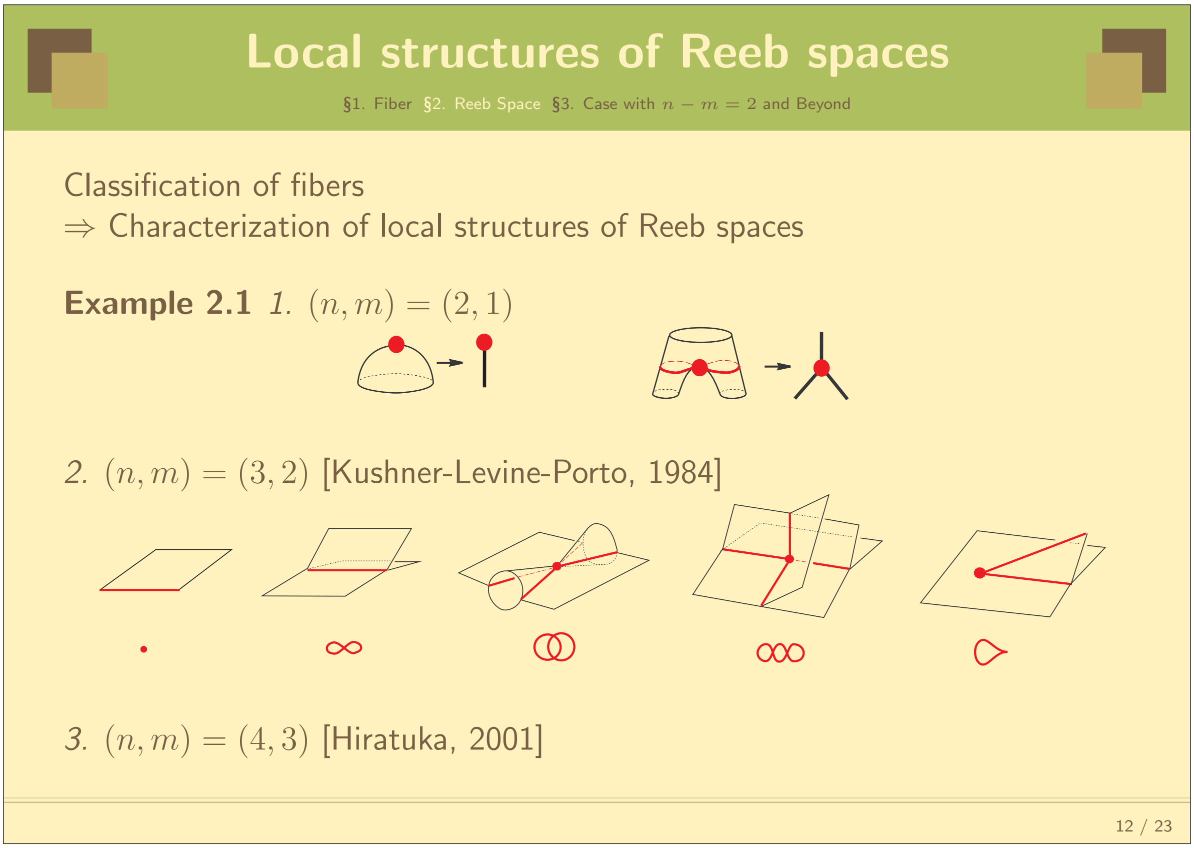
§2. Reeb Space



Reeb space

For a C^∞ map $f : N^n \to M^m$, $n > m$, the space R_f obtained by contracting each connected component of a fiber to a point is called the **Reeb space** of f [Edelsbrunner–Harer–Patel, 2008].



Local structures of Reeb spaces

Classification of fibers
$\Rightarrow$ Characterization of local structures of Reeb spaces

Example 2.1 *1.* $(n,m)=(2,1)$

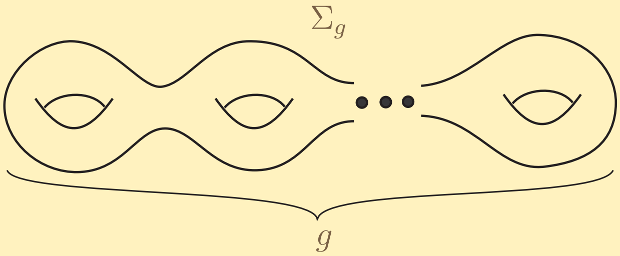
2. $(n,m)=(3,2)$ [Kushner-Levine-Porto, 1984]

3. $(n,m)=(4,3)$ [Hiratuka, 2001]

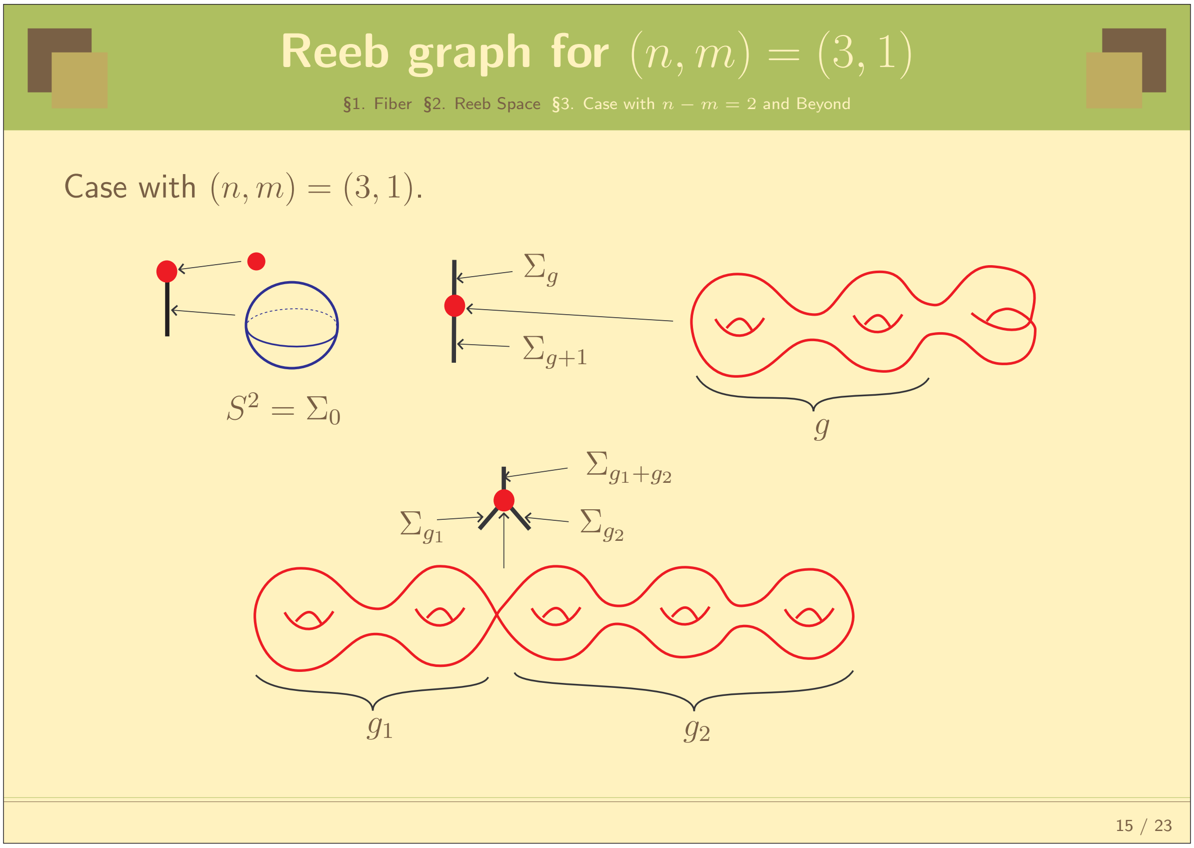
§3. Case with $n-m=2$ and Beyond

Regular fibers for $n-m=2$

How about the case $n-m=2$?
We assume the domain N^n is orientable, compact, and w/o boundary.
Regular fibers are **closed orientable surfaces**.

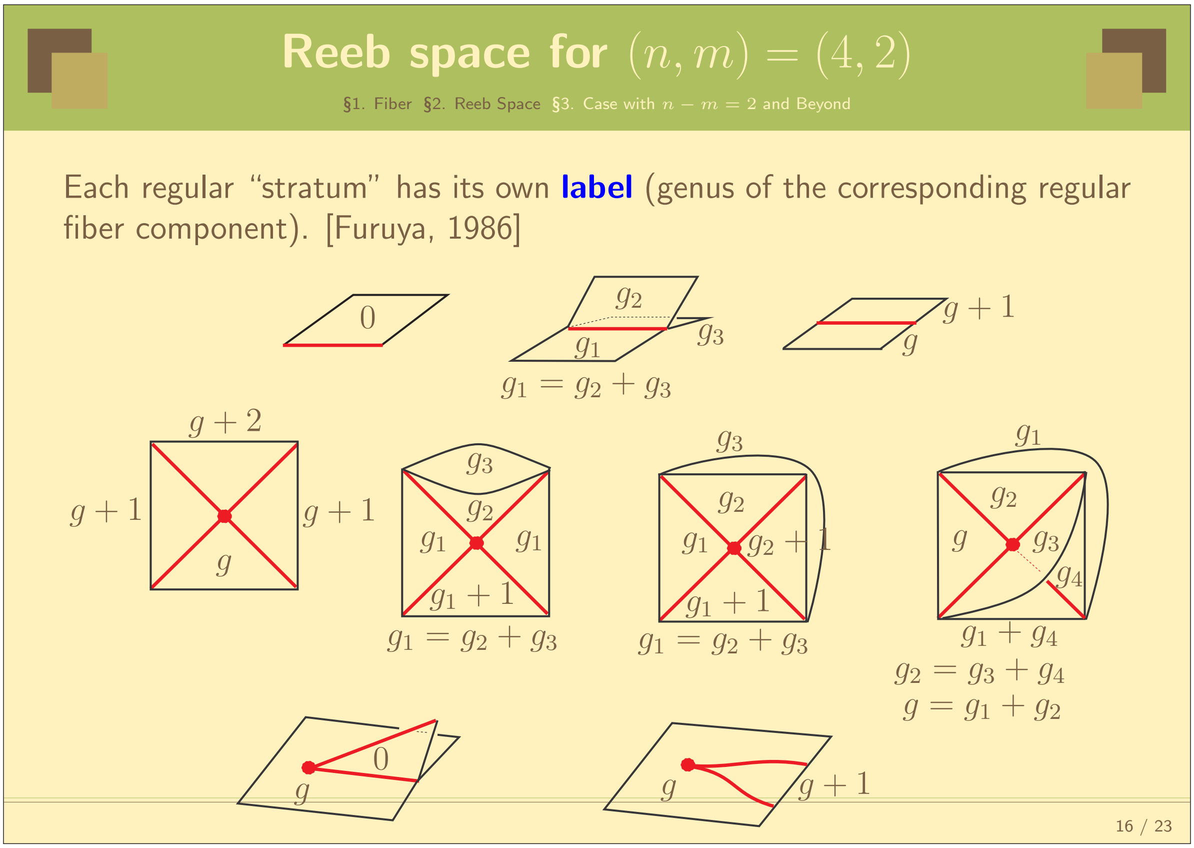


Σ_g : closed orientable surface of genus g



Reeb graph for $(n, m) = (3, 1)$

Case with $(n, m) = (3, 1)$.

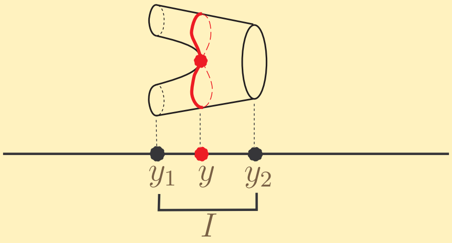


Reeb space for $(n, m) = (4, 2)$

Each regular "stratum" has its own **label** (genus of the corresponding regular fiber component). [Furuya, 1986]

0-th Homology

Reeb space describes the **connected components of fibers** and their "adjacencies".

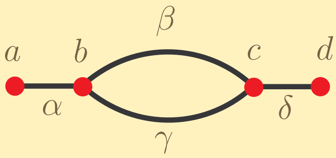


$$\begin{tikzcd}
H_0(f^{-1}(y_1)) \arrow[r] \arrow[d, "\cong"'] & H_0(f^{-1}(I)) \arrow[d, "\cong"'] & H_0(f^{-1}(y_2)) \arrow[l] \arrow[d, "\cong"'] \\
\mathbf{Z} \oplus \mathbf{Z} & H_0(f^{-1}(y)) \arrow[d, "\cong"'] & \mathbf{Z} \\
 & \mathbf{Z} &
\end{tikzcd}$$

Reeb diagram

This idea can be extended to homology groups of any dimension (or homotopy groups, if you want).

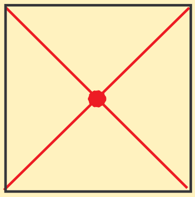
$$H_i(f^{-1}(y_1)) \to H_i(f^{-1}(y)) \leftarrow H_i(f^{-1}(y_2))$$



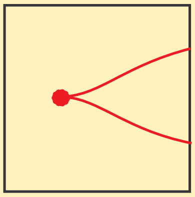
$$\begin{tikzcd}
 & & & H_i(\beta) \arrow[dl] \arrow[dr] & & & \\
H_i(a) & H_i(\alpha) \arrow[l] \arrow[r] & H_i(b) & & H_i(c) & H_i(\delta) \arrow[l] \arrow[r] & H_i(d) \\
 & & & H_i(\gamma) \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

$\Longrightarrow$ Notion of **Reeb diagram**.

Category ?



$$\begin{tikzcd}
 & & K_1 \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & H_2 \arrow[dr] & & H_1 \arrow[dl] & \\
K_2 \arrow[ur] \arrow[rr] \arrow[dr] & & G & & K_4 \arrow[ul] \arrow[ll] \arrow[dl] \\
 & H_3 \arrow[ur] & & H_4 \arrow[ul] & \\
 & & K_3 \arrow[ul] \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$



$$\begin{tikzcd}
 & & H_1 \arrow[dl] & \\
K_2 \arrow[urr] \arrow[r] \arrow[drr] & G & & K_1 \arrow[ul] \arrow[ll] \arrow[dl] \\
 & & H_2 \arrow[ul] &
\end{tikzcd}$$

In a certain **categorical formulation** of a Reeb space, this can be considered to be a **functor**.
Or any formulation using the notion of **sheaves**?

A formulation

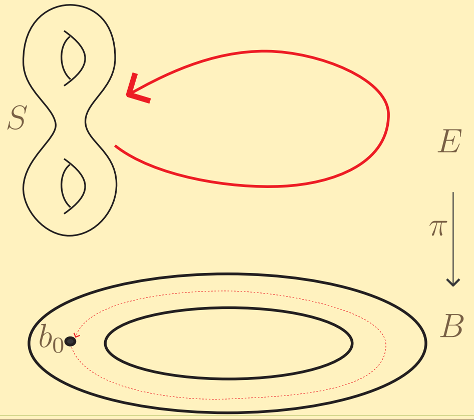
Reeb category
Object: each stratum (or simplex) of R_f
Morphism : $\tau \prec \sigma \Longleftrightarrow \tau \leftarrow \sigma$

Then, **Reeb diagram** is a **covariant functor** from Reeb category to the category of groups.

Remark 3.1 This makes sense if each stratum is contractible.

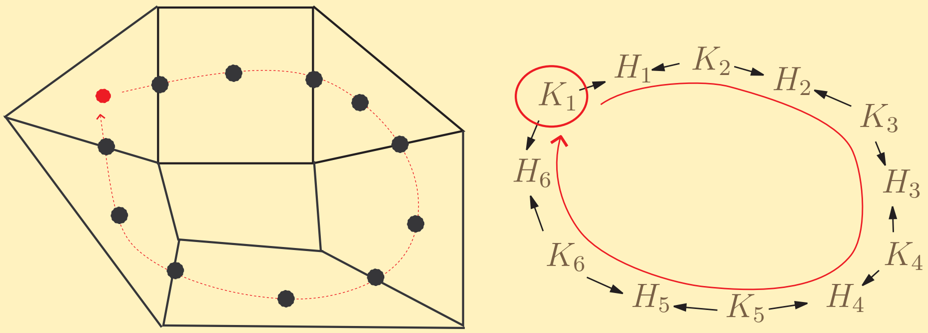
Monodromy

If a stratum is not contractible, we need to consider **monodromy**.
Suppose $\pi : E \to B$ is a C^∞ fiber bundle with fiber S.
$\mathrm{MCG}(S) = \pi_0(\mathrm{Diff}_+(S))$ **mapping class group**
Associated to π is the **monodromy** $\pi_1(B, b_0) \to \mathrm{MCG}(S)$.
This measures the "twist" of the fibers along a loop in the base B.



Monodromy in Reeb diagram

Given a generic map $f : N^n \to M^m$, we can subdivide M^m (or the Reeb space R_f) so that each stratum is contractible.
In this case, the **monodromy is hidden** in the Reeb diagram.



$$\pi_1(B, b_0) \to \mathrm{MCG}(S) \to \mathrm{Aut}(H_*(S))$$

Problem 3.2 *Formulate all these, including monodromy! Sheaf? Category theory? How to compute Reeb diagram and/or monodromy?*

Ending

Thank you!

「マス・フォア・インダストリ研究」シリーズ刊行にあたり

本シリーズは，平成23年4月に設立された九州大学マス・フォア・インダストリ研究所(IMI)が，平成25年4月に共同利用・共同研究拠点「産業数学の先進的・基礎的共同研究拠点」として，文部科学大臣より認定を受けたことにともない刊行するものである．本シリーズでは，主として，マス・フォア・インダストリに関する研究集会の会議録，共同研究の成果報告等を出版する．各巻はマス・フォア・インダストリの最新の研究成果に加え，その新たな視点からのサーベイ及びレビューなども収録し，マス・フォア・インダストリの展開に資するものとする．

平成30年10月
マス・フォア・インダストリ研究所
所長 佐伯　修

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マス・フォア・インダストリ研究　No.1	穴田　啓晃 安田　貴徳 Xavier Dahan 櫻井　幸一	Functional Encryption as a Social Infrastructure and Its Realization by Elliptic Curves and Lattices	26 February 2015
マス・フォア・インダストリ研究　No.2	滝口　孝志 藤原　宏志	Collaboration Between Theory and Practice in Inverse Problems	12 March 2015
マス・フォア・インダストリ研究　No.3	筧　三郎	非線形数理モデルの諸相：連続，離散，超離散，その先 (Various aspects of nonlinear mathematical models : continuous, discrete, ultra-discrete, and beyond)	24 March 2015
マス・フォア・インダストリ研究 No.4	穴田　啓晃 安田　貴徳 櫻井　幸一 寺西　勇	Next-generation Cryptography for Privacy Protection and Decentralized Control and Mathematical Structures to Support Techniques	29 January 2016
マス・フォア・インダストリ研究 No.5	藤原　宏志 滝口　孝志	Mathematical Backgrounds and Future Progress of Practical Inverse Problems	1 March 2016
マス・フォア・インダストリ研究 No.6	松谷　茂樹 佐伯　修 中川　淳一 上坂　正晃 濵田　裕康	結晶のらせん転位の数理	10 January 2017
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マス・フォア・インダストリ研究 No.9	阿部　拓郎 高島　克幸 縫田　光司 安田　雅哉	代数的手法による数理暗号解析 Workshop on analysis of mathematical cryptography via algebraic methods	1 March 2018
マス・フォア・インダストリ研究 No.10	阿部　拓郎 落合　啓之 高島　克幸 縫田　光司 安田　雅哉	量子情報社会に向けた数理的アプローチ Mathematical approach for quantum information society	26 December 2018

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マス・フォア・インダストリ研究 No.11	松谷　茂樹 佐伯　修 中川　淳一 濵田　裕康 上坂　正晃	結晶転位の先進数理解析 Advanced Mathematical Investigation for Dislocations	7 January 2019
マス・フォア・インダストリ研究 No.12	滝口　孝志	Non-destructive inspection for concrete structures and related topics	13 February 2019
マス・フォア・インダストリ研究 No.13	宇波　耕一 長野　智絵 吉岡　秀和 田上　大助 白井　朋之	数理農学における時系列データのモデル化と解析 Modeling and Analysis of Time Series Data in Math-Agro Sciences	28 February 2019
マス・フォア・インダストリ研究 No.14	佐久間　弘文 大津　元一 小嶋　泉 福本　康秀 山本　昌宏 納谷　昌之	ドレスト光子に関する基礎的数理研究	18 March 2019
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Institute of Mathematics for Industry
Kyushu University

九州大学マス・フォア・インダストリ研究所

〒819-0395 福岡市西区元岡744
https://www.imi.kyushu-u.ac.jp