Asymptotic behavior of compressible non-isothermal nematic liquid crystal flow in infinite layer

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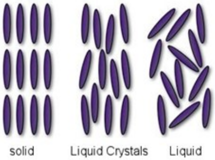
ワークショップ 数理・計算・データに基づく流体解析の最前線

1. Introduction

- Liquid crystals are in a state between solid and liquid.
- Nematic liquid cryatals can flow like a liquid but its molecules are oriented in a common direction as in solid.
- The flow of nematic liquid crystals is represented by coupling Navier-Stokes equation with direction equation based on Oseen-Frank energy density functional governing the motion of orientation of rod-like particles.



Simplified Ericksen-Leslie system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) &= 0, \\ \partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) &= \operatorname{div} \mathbb{S}_C, \\ \partial_t(\rho\theta) + \operatorname{div}(\rho\theta \boldsymbol{u}) - \kappa^* \Delta\theta &= \mathbb{S}_C : \nabla \boldsymbol{u}, \\ \partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} &= \tau^*(\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}). \end{cases}$$

- $\boldsymbol{u} = {}^{\top}(u^1(x,t), u^2(x,t), u^3(x,t))$: velocity, $\rho = \rho(x,t)$: density, $\theta = \theta(x,t)$: temperature and $\boldsymbol{d} = \boldsymbol{d}(x,t)$: director field for the averaged macroscopic molecular orientations (unknowns), $|\boldsymbol{d}| = 1$.
- $\mathbb{S}_C = \mathbb{S}_N - \eta^*(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} - \frac{1}{2}|\nabla \boldsymbol{d}|^2 \mathbb{I}) - P\mathbb{I}$,
  $\mathbb{S}_N = \mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\top}) + \mu'(\operatorname{div} \boldsymbol{u})\mathbb{I}$,
  $P = P(\rho, \theta)$: pressure,
  η^*, μ, μ^*, κ^*, τ^*: constants.
- $(\boldsymbol{u} \otimes \boldsymbol{u})_{ij} = u^i u^j$, $(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d})_{ij} = \partial_{x_i} \boldsymbol{d} \cdot \partial_{x_j} \boldsymbol{d}$, $\mathbb{S}_C : \nabla \boldsymbol{u} = \sum_{i,j} (\mathbb{S}_C)_{ij} \partial_{x_i} u^j$.

$x_3 = h;\ \theta = \theta_1^*$

Ω

$x_3 = 0;\ \theta = \theta_0^*$

- $x = (x', x_3)$
- Boundary condition:

$$\boldsymbol{u}|_{x_3=0,h} = \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{n}}|_{x_3=0,h} = \boldsymbol{0},$$

$$\theta|_{x_3=0} = \theta_0^*,\ \theta|_{x_3=h} = \theta_1^*.$$

2. Known Results

- Ericksen, Leslie (1950s – 1960s): continuum theory of liquid crystals
- Lin (1989): isothermal simplified model
- Lin, Liu (1995): global weak solutions for small initial data ($|\boldsymbol{d}| \neq 1$)
- Lin, Lin, Wang (2010): global weak solutions on bounded domain ($|\boldsymbol{d}| = 1$)
- Wang (2011): global well-posedness for small initial data in $BMO^{-1} \times BMO$ on $\mathbb{R}^n$

non-isothermal model

- Hieber, Prüss (2016): thermodinamically consistent modeling
- Feireisl, Frémond, Rocca, Schimperna (2012): global weak solutions for large initial data in bounded domain ($|\boldsymbol{d}| \neq 1$)
- Guo, Xi, Xie (2017): global well-posedness in $\mathbb{R}^3$, initial data close to steady state

Aim

To study the liquid crystals flow with thermal effect in infinite layer.

Non dimensionalized form:

(*)

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\
\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \operatorname{div}\big(\nu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top) + \nu'(\operatorname{div} \boldsymbol{u})\mathbb{I} \\
\qquad -\eta(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} - \frac{1}{2}|\nabla \boldsymbol{d}|^2 \mathbb{I}) - P\mathbb{I}\big), \\
\partial_t(\rho\theta) + \operatorname{div}(\rho\theta \boldsymbol{u}) - \kappa\Delta\theta = \beta\{\nu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top) + \nu'(\operatorname{div} \boldsymbol{u})\mathbb{I} \\
\qquad -\eta(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d} - \frac{1}{2}|\nabla \boldsymbol{d}|^2 \mathbb{I}) - P\mathbb{I}\} : \nabla \boldsymbol{u}, \\
\partial_t \boldsymbol{d} - \boldsymbol{u} \cdot \nabla \boldsymbol{d} = \tau(\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}),
\end{cases}
$$

$$
\boldsymbol{u}|_{x_3=0,1} = 0,
$$

$$
\theta|_{x_3=0} = 1, \ \theta|_{x_3=1} = \frac{\theta_1^*}{\theta_0^*},
$$

$$
\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{n}}|_{x_3=0,1} = \boldsymbol{0},
$$

$$
\rho \to \rho_s \quad (|x'| \to \infty).
$$

If $\alpha := \frac{\theta_0^*}{\theta_1^*} - 1$ is sufficiently small, then (*) has a stationary solution $u_s = (\rho_s, \boldsymbol{u}_s, \theta_s, \boldsymbol{d}_s)$:

$$\boldsymbol{u}_s = \mathbf{0},$$

$$\theta_s = \theta_s(x_3) = \alpha x_3 + 1,$$

$$\rho_s = \rho_s(x_3) = 1 + \phi(\alpha, x_3),$$

$$\phi(\alpha, x_3) = \frac{\partial_\theta^2 P(1,1)}{\partial_\rho P(1,1)} \alpha (x_3 - \frac{1}{2}) + \mathcal{O}(\alpha^2) \quad (\alpha \to 0),$$

$$\boldsymbol{d}_s = \boldsymbol{d}^* \text{ (constant)}$$

• perturbation $\phi = \rho - \rho_s$, $\tilde{\theta} = \theta - \theta_s$, $\tilde{\boldsymbol{d}} = \boldsymbol{d} - \boldsymbol{d}^*$

$$(**) \qquad \left\{ \begin{array}{rcl} \partial_t \phi + \mathrm{div}\,(\rho_s \boldsymbol{u}) &=& f, \\ \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} - (\nu + \nu') \nabla \mathrm{div}\, \boldsymbol{u} + \nabla \theta + \nabla \phi &=& \boldsymbol{g}, \\ \partial_t \theta - \kappa \Delta \theta + \beta (P(\rho_s, \theta_s) \mathrm{div}\, \boldsymbol{u}) &=& h, \\ \partial_t \boldsymbol{d} - \tau \Delta \boldsymbol{d} &=& \boldsymbol{k}, \end{array} \right.$$

where f, $\boldsymbol{g}$, h, $\boldsymbol{k}$ are nonlinear terms.

• Boundary conditions:

$$\boldsymbol{u}|_{x_3=0,1} = \theta|_{x_3=0,1} = 0, \ \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{n}}|_{x_3=0,1} = \boldsymbol{0}.$$

• Initial value:

$$u(0,x) = u_0 = (\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{d}_0).$$

Theorem 1

Suppose $\|\rho_s - 1\|_{H^6} \leq \exists\delta_1$, $\|\rho_s\theta_s - 1\|_{H^6} \leq \exists\delta_2$, $\|\theta_s - 1\|_{H^6} \leq \exists\delta_3$. If u_0 satisfies some compatibility conditions and $\|u_0\|_{(H^3)^3\times H^4} \ll 1$, then there exists a global unique solution $u(t)$ of (**) which satisfies

$$\|u(t)\|_{(H^3)^3\times H^4} \leq C\|u_0\|_{(H^3)^3\times H^4}.$$

Theorem 2

In addition to the assumption of Theorem 1, we assume that $u_0 \in (L^1(\Omega))^4$, then the solution $u(t)$ satisfies the following estimates:
(i)

$$\|\partial_{x_3}^{l}\partial_{x'}^{l'}u(t)\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

for $0 \leq l,\ |l'| \leq 1.$ Furthermore,

$$\|\partial_{x_3}^{l}\partial_{x'}^{l'}(\boldsymbol{u},\theta)(t)\|_{L^2(\Omega)} \leq C(1+t)^{-1}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

$$\|\partial_{x_3}^{l}\partial_{x'}^{l'}\boldsymbol{d}(t)\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{1}{2}-\frac{l+|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)}$$

for $0 \leq l,\ |l'| \leq 1,\ l+|l'| \leq 1.$

(ii)

$$\|u(t) - \tilde{\sigma}_0(t)\|_{L^2(\Omega)} = o(t^{-\frac{1}{2}}) \text{ as } t \to \infty,$$

where

$$\tilde{\sigma}_0(t) = {}^\top(\tilde{\phi}_{low}(t), \mathbf{0}, 0, \tilde{\boldsymbol{d}}_{low}(t)),$$

$$\tilde{\phi}_{low}(t) = \alpha_0 G_0(x', t), \ \tilde{\boldsymbol{d}}_{low} = {}^\top(\beta_1 G_1(x', t), \beta_2 G_2(x', t), \beta_3 G_3(x', t)),$$

$$\alpha_0 = \left(\int_\Omega \phi_0(y)\, dy\right) \frac{1}{\partial_\rho P(\rho_s, \theta_s)}, \ \beta_j = \int_\Omega d_0^j(y)\, dy + \int_0^\infty \int_\Omega k^j(\boldsymbol{u}, \boldsymbol{d})(y, s)\, dyds,$$

$$G_j(t) = (4\pi\kappa_j t)^{-1} \mathrm{e}^{-\frac{|x'|^2}{4\kappa_j t}} \quad (j = 0, 1, 2, 3, \ \kappa_j > 0).$$

Remark

Since $\|G_j(x', t)\|_{L^2} = Ct^{-\frac{1}{2}}$, the estimate in (ii) is meaningful.

4. Outline of Proof

Decay estimates

The Fourier transform of (**) in x' variable is written as

$$\partial_t \hat{u} + \hat{L}_{\xi'} \hat{u} = \hat{F}(u).$$

$\hat{L}_{\xi'}$ is the linearized operator which has the form

$$\hat{L}_{\xi'} = \hat{L}_0 + \sum_{j=1,2} \xi_j \hat{L}_j^{(1)} + \sum_{j,k=1,2} \xi_j \xi_k \hat{L}_{j,k}^{(2)}$$

where

$$\hat{L}_0 = \begin{pmatrix} 0 & {}^\top\mathbf{0} & \partial_{x_3}(\rho_s \cdot) & 0 & {}^\top\mathbf{0} \\ \mathbf{0} & -\frac{\nu}{\rho_s}\partial_{x_3}^2 I_2 & \mathbf{0} & \mathbf{0} & O \\ \frac{1}{\rho_s}\partial_{x_3}(\tilde{P}_\rho \cdot) & {}^\top\mathbf{0} & -\frac{2\nu+\nu'}{\rho_s}\partial_{x_3}^2 & \frac{1}{\rho_s}\partial_{x_3}(\tilde{P}_\theta \cdot) & {}^\top\mathbf{0} \\ 0 & {}^\top\mathbf{0} & \frac{\beta \tilde{P}}{\rho_s}\partial_{x_3} + \alpha & -\frac{\kappa}{\rho_s}\partial_{x_3}^2 & {}^\top\mathbf{0} \\ \mathbf{0} & O & \mathbf{0} & \mathbf{0} & -\tau \partial_{x_3}^2 I_3 \end{pmatrix},$$

$$\hat{L}_j^{(1)} = \begin{pmatrix} 0 & i\rho_s {}^\top\boldsymbol{e}'_j & 0 & 0 & {}^\top\mathbf{0} \\ i\frac{\tilde{P}_\rho}{\rho_s}\boldsymbol{e}'_j & O & -i\frac{\nu+\nu'}{\rho_s}\boldsymbol{e}'_j\partial_{x_3} & i\frac{\tilde{P}_\theta}{\rho_s}\boldsymbol{e}'_j & {}^\top\mathbf{0} \\ 0 & -i\frac{\nu+\nu'}{\rho_s}{}^\top\boldsymbol{e}'_j\partial_{x_3} & 0 & 0 & {}^\top\mathbf{0} \\ 0 & i\frac{\beta\tilde{P}}{\rho_s}{}^\top\boldsymbol{e}'_j & 0 & 0 & {}^\top\mathbf{0} \\ \mathbf{0} & O & \mathbf{0} & \mathbf{0} & O \end{pmatrix}$$

and

$$\hat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & {}^\top\mathbf{0} & 0 & 0 & {}^\top\mathbf{0} \\ 0 & \frac{\nu}{\rho_s}\delta_{jk}I_2 + \frac{\nu+\nu'}{\rho_s}\boldsymbol{e}'_j {}^\top\boldsymbol{e}'_k & \mathbf{0} & \mathbf{0} & O \\ 0 & {}^\top\mathbf{0} & \frac{\nu}{\rho_s}\delta_{jk} & 0 & {}^\top\mathbf{0} \\ 0 & {}^\top\mathbf{0} & 0 & \frac{\kappa}{\rho_s}\delta_{jk} & {}^\top\mathbf{0} \\ \mathbf{0} & O & \mathbf{0} & \mathbf{0} & \tau\delta_{jk}I_3 \end{pmatrix}.$$

Proposition 1

There exist positive constants $\tilde{c}_0$ and $\tilde{c}_1$ such that

$$\rho(-\hat{L}_0) \supset \Sigma \backslash \{0\},$$

where $\Sigma = \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -\tilde{c}_1 |\mathrm{Im}\,\lambda|^2 - \tilde{c}_0\}$. The eigenspace for eigenvalue 0 are spanned by ${}^\top(\frac{1}{\partial_\rho P(\rho_s,\theta_s)}, \mathbf{0}, 0, 0, \mathbf{0})$ and ${}^\top(0, \mathbf{0}, 0, 0, \boldsymbol{e}_j)$, $j = 1, 2, 3$.

We set the eigenprojection for eigenvalue 0 of $-\hat{L}_0$ as $\hat{\Pi}^{(0)} = \hat{\Pi}_1^{(0)} + \hat{\Pi}_2^{(0)}$ with

$$\hat{\Pi}_1^{(0)} \hat{u} = a \langle \hat{\phi} \rangle \begin{pmatrix} \frac{1}{\partial_\rho P(\rho_s,\theta_s)} \\ \mathbf{0} \\ 0 \\ 0 \\ \mathbf{0} \end{pmatrix},\ \hat{\Pi}_2^{(0)} = \sum_{j=1}^{3} \hat{\Pi}_{2,j}^{(0)},\ \hat{\Pi}_{2,j}^{(0)} \hat{u} = \langle \hat{d}^j \rangle \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ 0 \\ \boldsymbol{e}_j \end{pmatrix}, j = 1, 2, 3$$

for $\hat{u} = {}^\top(\hat{\phi}, \hat{\boldsymbol{u}}', \hat{u^3}, \hat{\theta}, \hat{\boldsymbol{d}})$. Here $a = (\int_0^1 \frac{1}{\partial_\rho P(\rho_s,\theta_s)}\, dx_3)^{-1}$, $\langle \cdot \rangle = \int_0^1 \cdot\, dx_3$.

Proposition 2

There exists positive number $r_0 = r_0(\tilde{c}_0, \tilde{c}_1)$ such that for $|\xi'| \leq r_0$,

$$\sigma(-\hat{L}_{\xi'}) \cap \{\lambda \in \mathbb{C};\ |\lambda| \leq \tilde{c}_0\} = \{\lambda_0(\xi')\} \cup \{-\tau|\xi'|^2\}.$$

Here $\lambda_0(\xi')$ is a simple eigenvalue of $-L_{\xi'}$:

$$\lambda_0(\xi') = -\frac{ab}{2\nu}|\xi'|^2 + \mathcal{O}(|\xi'|^3) \text{ as } |\xi'| \to 0,$$

where $b = \int_0^1 (x_3^2 - x_3)\rho_s(x_3)\,dx_3$ and $-\tau|\xi'|^2$ is a semisimple eigenvalue. The eigenprojection $\hat{\Pi}_1(\xi')$ for $\lambda_0(\xi')$ is given by

$$\hat{\Pi}_1(\xi') = \hat{\Pi}_1^{(0)} + \hat{\Pi}_1^{(1)}(\xi'),$$

where $|\hat{\Pi}_1^{(1)}(\xi')u|_{H^k} \leq C_k|u|_{L^2}$ for $k = 0, 1, 2, \cdots$. The eigenprojection $\hat{\Pi}_2(\xi')$ for $-\tau|\xi'|^2$ is given by

$$\hat{\Pi}_2(\xi') = \hat{\Pi}_2^{(0)}$$

Decompose $\partial_t \hat{u} + \hat{L}_{\xi'} \hat{u} = \hat{F}(u)$ into high-frequency part and low-frequency part by using

$$\begin{cases} \chi_0(\xi') = 1 \ \ (|\xi'| \le R), \\ \chi_0(\xi') = 0 \ \ (|\xi'| > R), \end{cases}$$

$$\chi_\infty(\xi') = 1 - \chi_0(\xi')$$

for some positive constant $R < r_0$.

Let $\hat{\Pi}(\xi')$ be the total eigenprojection associated with eigenvalues $\lambda_0(\xi')$ and $-\tau|\xi'|^2$ which is given by $\hat{\Pi}(\xi') = \hat{\Pi}_1(\xi') + \hat{\Pi}_2(\xi')$.

$$
\begin{cases}
\partial_t \hat{U}_0 + \hat{L}_{\xi'} \hat{U}_0 = \hat{P}_0 \hat{F}(u), \\
\partial_t \hat{U}_{0,\infty} + \hat{L}_{\xi'} \hat{U}_{0,\infty} = \hat{P}_{0,\infty} \hat{F}(u), \\
\partial_t \hat{\hat{U}}_\infty + \hat{L}_{\xi'} \hat{\hat{U}}_\infty = \hat{\hat{P}}_\infty \hat{F}(u),
\end{cases}
$$

where $\hat{P}_0 = \chi_0 \hat{\Pi}(\xi')$, $\hat{P}_{0,\infty} = \chi_0 (I - \hat{\Pi}(\xi'))$ and $\hat{\hat{P}}_\infty = \chi_\infty$.

Proposition 3

We have the following estimate for high frequency part.

$$
\|U_\infty(t)\|_{H^3} \leq C(1+t)^{-1} (\|U_\infty(0)\|_{H^3} + \|\nabla \boldsymbol{d}_\infty(0)\|_{H^3}).
$$

Here $U_\infty(t) = \mathcal{F}^{-1}(\hat{U}_{0,\infty}(t) + \hat{U}_\infty(t))$.

$\hat{P}_0$ - part

Decompose $\hat{U}_0(t)$ into eigenspace of eigenvalue 0 of $-\hat{L}_0$ and its complementary space.

$$\hat{U}_0(t) = \hat{\sigma_0}(t) + \hat{U}_1(t),$$

where $\hat{\sigma}_0 = \hat{\Pi}^{(0)}\hat{U}_0 = \hat{\Pi}^{(0)}(\chi_0\hat{\Pi}(\xi')\hat{u})$ and
$\hat{U}_1 = (I - \hat{\Pi}^{(0)})\hat{U}_0 = (I - \hat{\Pi}^{(0)})(\chi_0\hat{\Pi}(\xi')\hat{u})$. We set

$$\sigma_0 = \mathcal{F}^{-1}\hat{\sigma_0},\ U_1 = \mathcal{F}^{-1}\hat{U}_1.$$

We also define ϕ_{low} and $\boldsymbol{d}_{low}$ by

$$\sigma_0 = {}^{\top}(\phi_{low}, \mathbf{0}, 0, \boldsymbol{d}_{low}).$$

Lemma 4

The following estimates hold for multi index l_1' and l_2':

$$\|\partial_{x'}^{l_1'}\partial_{x_3}^{l} U_0\|_2 \le C_{l_2'} C_l \|\partial_{x'}^{l_2'} U_0\|_2 \quad (0 \le l_2' \le l_1'),$$

$$\|U_0\|_\infty \le \|U_0\|_{H^2} \le C\{\|\phi_{low}\|_2 + \|\boldsymbol{d}_{low}\|_2\},$$

$$\|\partial_{x'}^{l'}\partial_{x_3}^{l} U_1\|_\infty \le C\|\partial_{x'}\sigma_0\|_2.$$

We set

$$M_1(t) = \sup_{0\le s\le t} \sum_{|l'|=0}^{1} (1+s)^{\frac{1}{2}+\frac{|l'|}{2}} \|\partial_{x'}^{l'}\sigma_0(s)\|_{L^2},$$

$$M_\infty(t) = \sup_{0\le \tau\le t} (1+\tau)\|U_\infty(\tau)\|_{H^3}$$

and

$$M(t) = M_1(t) + M_\infty(t).$$

We also set

$$P_0 = \mathcal{F}^{-1}\hat{P}_0\mathcal{F},\ P_{0,\infty} = \mathcal{F}^{-1}\hat{P}_{0,\infty}\mathcal{F},\ P_\infty = \mathcal{F}^{-1}\hat{P}_\infty\mathcal{F}.$$

Proposition 5

For the eigenspace of eigenvelue 0 of $-\hat{L}_0$ we have the following decay estimate:

$$\|\partial_{x'}^{l'}\partial_{x_3}^{l}\sigma_0(t)\|_2 \le C\|\partial_{x'}^{l'}\sigma_0(t)\|_2 \le C(1+t)^{-\frac{|l'|}{2}-\frac{1}{2}}\|\sigma_0(0)\|_1.$$

Proof We prove for $\boldsymbol{d}_{low}$.

$$\partial_t\hat{\boldsymbol{d}}_{low} + \tau|\xi'|^2\hat{\boldsymbol{d}}_{low} = \chi_0\langle\hat{\boldsymbol{k}}\rangle,$$
$$\hat{\boldsymbol{d}}_{low}|_{t=0} = \chi_0\langle\hat{\boldsymbol{d}}_0\rangle.$$

From this we see that

$$\hat{\boldsymbol{d}}_{low}(\xi',t) = \mathrm{e}^{-\tau|\xi'|^2 t}\chi_0\langle\hat{\boldsymbol{d}}_0\rangle(\xi') + \int_0^t \mathrm{e}^{-\tau|\xi'|^2(t-s)}\chi_0\langle\hat{\boldsymbol{k}}\rangle(\xi',s)\,ds$$
$$=: I_1 + I_2.$$

$$
\begin{aligned}
\|\partial_{x'}^{l'}\mathcal{F}^{-1}I_1\|_2^2 &= \|(i\xi')^{l'}\mathrm{e}^{-\tau|\xi'|^2t}\chi_0\langle\hat{\boldsymbol{d}}_0\rangle\|_2^2 \\
&\le C\int_{|\xi'|\le R}|\xi'|^{2|l'|}\mathrm{e}^{-2\tau|\xi'|^2t}\,d\xi'\left(\sup_{|\xi'|\le R}|\langle\hat{\boldsymbol{d}}_0\rangle(\xi')|\right)^2 \\
&\le C(\tau t)^{-|l'|-1}\left(\sup_{|\xi'|\le R}|\langle\hat{\boldsymbol{d}}_0\rangle(\xi')|\right)^2 \\
&\le C(\tau t)^{-|l'|-1}\|\sigma_0(0)\|_1^2.
\end{aligned}
$$

So we have

$$
\|\partial_{x'}^{l'}\mathcal{F}^{-1}I_1\|_2 \le C(\tau t)^{-\frac{|l'|}{2}-\frac{1}{2}}\|\sigma_0(0)\|_1.
$$

On the other hand,

$$
\begin{aligned}
\|\partial_{x'}^{l'}\mathcal{F}^{-1}I_1\|_2^2 &\le C\int_{\mathbb{R}^2}\chi_0|\xi'|^{2|l'|}\mathrm{e}^{-2\tau\chi_0|\xi'|^2t}\,d\xi'\left(\sup_{|\xi'|\le R}|\langle\hat{\boldsymbol{d}}_0\rangle(\xi')|\right)^2\\
&\le C\int_{|\xi'|\le R}R^{2|l'|}\mathrm{e}^{-2\tau\chi_0|\xi'|^2t}\,d\xi'\|\sigma_0(0)\|_1^2\\
&\le C_R\|\sigma_0(0)\|_1^2.
\end{aligned}
$$

Therefore we get

$$
\|\partial_{x'}^{l'}\mathcal{F}^{-1}I_1\|_2 \le C(1+t)^{-\frac{|l'|}{2}-\frac{1}{2}}\|\sigma_0(0)\|_1.
$$

In a similar manner, we have

$$
\left\|\partial_{x'}^{l'}\mathcal{F}^{-1}I_2\right\|_2 \le C\int_0^t(1+t-s)^{-\frac{|l'|}{2}-\frac{1}{2}}\|\boldsymbol{k}(s)\|_1\,ds.
$$

Nonlinear term $\boldsymbol{k}$ is written as

$$\begin{aligned}\boldsymbol{k} &= -\boldsymbol{u}\cdot\nabla\boldsymbol{d} + \tau|\nabla\boldsymbol{d}|^2(\boldsymbol{d}+\boldsymbol{d}^*)\\ &= -(P_0\boldsymbol{u}+P_\infty\boldsymbol{u})\cdot\nabla(P_0\boldsymbol{d}+P_\infty\boldsymbol{d})\\ &\quad + \tau|\nabla(P_0\boldsymbol{d}+P_\infty\boldsymbol{d})|^2(P_0\boldsymbol{d}+P_\infty\boldsymbol{d}+\boldsymbol{d}^*)\\ &= -P_0\boldsymbol{u}\cdot\nabla P_0\boldsymbol{d} + \tau|\nabla P_0\boldsymbol{d}|^2(P_0\boldsymbol{d}+\boldsymbol{d}^*) + \mathcal{K}_\infty.\end{aligned}$$

Here $\mathcal{K}_\infty$ is nonlinear terms which include $P_\infty\boldsymbol{u}$ or $P_\infty\boldsymbol{d}$. Then

$$\begin{aligned}\|\boldsymbol{k}\|_1 &\le C\{\|U_1\|_2\|\nabla U_0\|_2 + \|\nabla U_0\|_2^2(\|U_0\|_{H^2}+|\boldsymbol{d}^*|) + \|\mathcal{K}_\infty\|_1\}\\ &\le C\|\partial_{x'}\sigma_0\|_2^2 + \|\partial_{x'}\sigma_0\|_2^2(\|\sigma_0\|_2+1) + \|\mathcal{K}_\infty\|_1\}.\end{aligned}$$

Here we use Lemma 4.

Hence

$$
\begin{aligned}
&\|\partial_{x'}^{l'} \boldsymbol{d}_{low}(t)\|_2 \\
&\quad \le C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}} \|\sigma_0(0)\|_1 \\
&\quad + C\int_0^t (1+t-s)^{-\frac{1}{2}-\frac{|l'|}{2}} (1+s)^{-2}\, ds\, M(t)^2 \\
&\quad \le C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}} \|\sigma_0(0)\|_1 + C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}} M(t)^2,
\end{aligned}
$$

so we have

$$
(1+t)^{\frac{1}{2}+\frac{|l'|}{2}} \|\partial_{x'}^{l'} \boldsymbol{d}_{low}(t)\|_2 \le C(\|\sigma_0(0)\|_1 + M(t)^2).
$$

Asymptotic behavior

It suffices to derive the asymptotic leading part of $\phi_{low}(t)$ and $\boldsymbol{d}_{low}(t)$. We here consider $\boldsymbol{d}_{low}(t)$.

$$\hat{\boldsymbol{d}}_{low}(\xi', t) = \mathrm{e}^{-\tau|\xi'|^2 t}\chi_0 \langle \hat{\boldsymbol{d}}_0 \rangle(\xi') + \int_0^t \mathrm{e}^{-\tau|\xi'|^2(t-s)}\chi_0 \langle \hat{\boldsymbol{k}} \rangle(\xi', s)\, ds.$$

Since $\int_0^\infty \int_\Omega k^j(\boldsymbol{u}, \boldsymbol{d})(y, s)\, dyds = \int_0^\infty \langle \hat{k}^j \rangle(0, s)\, ds$, we investigate

$$\begin{aligned} &\hat{d}^j_{low}(\xi', t) - \left(\langle \hat{d}^j_0 \rangle(0) + \int_0^\infty \langle \hat{k}^j \rangle(0, s)\, ds \right) \mathrm{e}^{-\tau|\xi'|^2 t} \\ &= \chi_0 (\langle \hat{d}^j_0 \rangle(\xi') - \langle \hat{d}^j_0 \rangle(0)) \mathrm{e}^{-\tau|\xi'|^2 t} + (\chi_0 - 1) \langle \hat{d}^j \rangle(0) \mathrm{e}^{-\tau|\xi'|^2 t} \\ &\quad + \int_0^t \mathrm{e}^{-\tau|\xi'|^2(t-s)} \chi_0 \langle \hat{k}^j \rangle(\xi', s)\, ds - \mathrm{e}^{-\tau|\xi'|^2 t} \int_0^\infty \langle \hat{k}^j \rangle(0, s)\, ds \end{aligned}$$

$$
\begin{aligned}
&= \chi_0(\hat{d}_0^j(\xi') - \hat{d}_0^j(0))\mathrm{e}^{-\tau|\xi'|^2 t} + (\chi_0 - 1)\langle \hat{d}^j\rangle(0)\mathrm{e}^{-\tau|\xi'|^2 t} \\
&\quad - \mathrm{e}^{-\tau|\xi'|^2 t}\int_{\frac{t}{2}}^{\infty} \langle \hat{k}^j\rangle(0,s)\,ds \\
&\quad + \int_0^{\frac{t}{2}} (\mathrm{e}^{-\tau|\xi'|^2(t-s)} - \mathrm{e}^{-\tau|\xi'|^2 t})\langle \hat{k}^j\rangle(0,s)\,ds \\
&\quad + \int_0^{\frac{t}{2}} \mathrm{e}^{-\tau|\xi'|^2(t-s)}(\chi_0\langle \hat{k}^j\rangle(\xi',s) - \langle \hat{k}^j\rangle(0,s))\,ds \\
&\quad + \int_{\frac{t}{2}}^{t} \mathrm{e}^{-\tau|\xi'|^2(t-s)}\chi_0\langle \hat{k^j}\rangle(\xi',s)\,ds \\
&=: I_0 + I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

We can see that

$$\begin{aligned}\|I_1\|_2^2 &= \int_{|\xi'|\geq R} |\langle \hat{d}^j \rangle(0)|^2 \mathrm{e}^{-2\tau|\xi'|^2 t}\, d\xi' \\ &\leq \mathrm{e}^{-2\tau R^2 t} \|\boldsymbol{d}_0\|_2^2.\end{aligned}$$

It is proved that $\|\boldsymbol{k}(s)\|_1 \leq C(1+s)^{-2}$ in proof of Proposition 5, hence we have

$$\begin{aligned}\|I_2\|_2 &\leq \|\mathrm{e}^{-\tau|\xi'|^2 t}\|_2 \int_{\frac{t}{2}}^{\infty} |\langle \hat{k}^j \rangle(0,s)|\, ds \\ &\leq \|\mathrm{e}^{-\tau|\xi'|^2 t}\|_2 \int_{\frac{t}{2}}^{\infty} \|k^j(\cdot,s)\|_1\, ds \\ &\leq C(1+t)^{-\frac{1}{2}} \int_{\frac{t}{2}}^{\infty} (1+s)^{-2}\, ds \\ &\leq C(1+t)^{-\frac{3}{2}}.\end{aligned}$$

For I_3, it can be seen that

$$\begin{aligned} e^{-\tau|\xi'|^2(t-s)} - e^{-\tau|\xi'|^2 t} &= \int_0^1 \frac{d}{d\tilde{\tau}}\left(e^{-\tau|\xi'|^2(t-\tilde{\tau}s)}\right) d\tilde{\tau} \\ &= \tau|\xi'|^2 s \int_0^1 e^{-\tau|\xi'|^2(t-\tilde{\tau}s)}\, d\tilde{\tau}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|I_3\|_2 &\le \int_0^1 \int_0^{\frac{t}{2}} \tau s \||\xi'|^2 e^{-\tau|\xi'|^2(t-\tilde{\tau}s)}\|_2 \|k^j(s)\|_1\, ds d\tilde{\tau} \\ &\le C \int_0^1 \int_0^{\frac{t}{2}} (1+t-\tilde{\tau}s)^{-\frac{3}{2}} (1+s)^{-1}\, ds d\tilde{\tau} \\ &\le C(1+t)^{-\frac{3}{2}} \int_0^1 \int_0^{\frac{t}{2}} (1+s)^{-1}\, ds d\tilde{\tau} \\ &\le C(1+t)^{-\frac{3}{2}} \log(1+t). \end{aligned}$$

By changing variables $\sqrt{\tau(t-s)}\xi' = \eta$, we see

$$\|I_4\|_2$$

$$= \int_0^{\frac{t}{2}} \left(\int_{\mathbb{R}^2} \mathrm{e}^{-2|\eta|^2} \left(\chi_0 \langle \hat{k}^j \rangle \left(\frac{\eta}{\sqrt{\tau(t-s)}}, s \right) - \langle \hat{k}^j \rangle (0,s) \right)^2 \frac{1}{\tau(t-s)} \, d\eta \right)^{\frac{1}{2}} ds$$

$$\leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}} \left\| \mathrm{e}^{-|\eta|^2} \left(\chi_0 \langle \hat{k}^j \rangle \left(\frac{\eta}{\sqrt{\tau(t-s)}}, s \right) - \langle \hat{k}^j \rangle (0,s) \right) \right\|_2 ds$$

$$\leq C t^{-\frac{1}{2}} \int_0^{\frac{t}{2}} J_t(s)^{\frac{1}{2}} \, ds,$$

where $J_t(s) = \int_{\mathbb{R}^2} \mathrm{e}^{-2|\eta|^2} \left| \chi_0 \langle \hat{k}^j \rangle \left(\frac{\eta}{\sqrt{\tau(t-s)}}, s \right) - \langle \hat{k}^j \rangle (0,s) \right|^2 d\eta$.

Since

$$\mathrm{e}^{-2|\eta|^2}\left|\chi_0\langle\hat{k}^j\rangle\left(\frac{\eta}{\sqrt{\tau(t-s)}},s\right)-\langle\hat{k}^j\rangle(0,s)\right|^2\le C\mathrm{e}^{-|\eta|^2}\|k^j(s)\|_1^2\in L^1_\eta$$

and

$$\mathrm{e}^{-2|\eta|^2}\left|\chi_0\langle\hat{k}^j\rangle\left(\frac{\eta}{\sqrt{\tau(t-s)}},s\right)-\langle\hat{k}^j\rangle(0,s)\right|^2\to 0\ \ (t\to\infty),$$

by the dominated convergence theorem we have $J_t(s)\to 0$ as $t\to 0$ for every s. Furthermore, since

$$\begin{aligned} J_t(s) &\le C\int_{\mathbb{R}^2}\mathrm{e}^{-2|\eta|^2}\,d\eta\,\|k^j(s)\|_1^2 \\ &\le C(1+s)^{-2}\in L^1_s \end{aligned}$$

for all t, we can apply the dominated convergence theorem again to have

$$\int_0^{\frac{t}{2}} J_t(s)\,ds\to 0\ \ (t\to\infty),$$

hence

$$t^{\frac{1}{2}}\|I_4\|_2\to 0\ \ (t\to\infty).$$

Similarly, we can estimate I_0 to see that $t^{\frac{1}{2}}\|I_0(t)\|_{L^2} \to 0$ as $t \to \infty$. Finally we obtain that

$$\begin{aligned}
\|I_5\|_2 &\le \int_t^{\frac{t}{2}} \|\mathrm{e}^{-\tau|\xi'|^2(t-s)}\|_{L^2}\|k^j(s)\|_1\, ds \\
&\le C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{1}{2}}\|k^j(s)\|_1\, ds \\
&\le C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{1}{2}}(1+s)^{-2}\, ds \\
&\le C(1+t)^{-2}[-(1+t-s)^{\frac{1}{2}}]_{\frac{t}{2}}^{t} \\
&\le C(1+t)^{-\frac{3}{2}}.
\end{aligned}$$

Thank you for your kind attention!